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ABSTRACT

This text is one of the sequence of textbooks produced for low achievers in the seventh and eighth grades by the School Mathematics Study Group (SMSG). There are eight texts in the sequence, of which this is the sixth. This set of volumes differs from the regular editions of SMSG junior high school texts in that very little reading is required. Concepts and processes are illustrated pictorially, and many exercises are included. Similarity of triangles is the focus of the first chapter (12) in this volume. The use of ratios and scale factors is introduced, and the computation of percentages by construction of parallel lines on a grid is developed. In chapter 13 the emphasis is on computation with rational numbers in both common fraction and decimal forms. In this context exponents are introduced. Chapter 14 deals with motion geometry and perpendicularity. (SD)

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SECONDARY SCHOOL MATHEMATICS

SPECIAL EDITION

Chapter 12. Similarity

Chapter 13. More About Rational Numbers

Chapter 14. Perpendiculars

STUDENT'S TEXT

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SE 019 488

SECONDARY SCHOOL MATHEMATICS

SPECIAL EDITION

Chapter 12. Similarity

Chapter 13. More About
Rational Numbers

Chapter 14. Perpendiculars

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TABLE OF CONTENTS

Page

Chapter 12 - Similarity 1
12-1. Similar Figures. 2
12-2. Similar Triangles. 9
12-3. Ratios and Scale Factors 15
12-4. How a Photo Enlarger Works ; Parallels and Similarity 22
12-5. How to Divide Up a Line Segment. 28
12-6. Ratios and Similar Triangles 34
12-7. Similar Triangles, Ratios, and Percent . . . 40
12-8. Solving Percent Problems 48
Pre-Test Exercises 53
Test 59
Check Your Memory: Self-Test. 66

Chapter 13 - More About Rational Numbers. 71
13-1. Another Look at Integers 74
13-2. Multiplying and Dividing with All the Integers 81
13-3. Another Look at Rational Numbers 84
13-4. Multiples. 89
13-5. Common Multiples 93
13-6. Adding Rational Numbers. 100
13-7. Subtracting Rational Numbers 107
13-8. Rational Numbers That Are Powers of Ten. . . 109
13-9. Decimals and Powers of Ten 114
13-10. Scientific Notation. 119
13-11. Rational Numbers on the Number Line. 127
13-12. Fractions, Decimals and Percents 133
Pre-Test Exercises 139
Test 145
Check Your Memory: Self-Test. 148

Table of Contents Continued

Page

Chapter 14 - Perpendiculars 153

14-1. Congruent Figures and Motions. 154

14-2. Congruence of a Figure with Itself 161

14-3. Right Angles and Perpendicular Lines 167

14-4. Sets of Equidistant Points in a Plane (Perpendicular Bisectors). 169

14-5. Circles and Perpendiculars 173

14-6. Triangles and Paper Folding. 178

Pre-Test Exercises 186

Test 191

Check Your Memory: Self-Test. 195

Chapter 12.

SIMILARITY

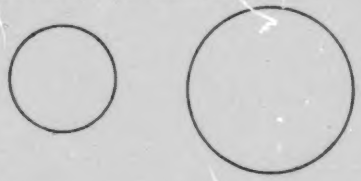
Chapter 12

SIMILARITY

Similar Figures

In general, we say that two figures are similar if they have the same shape but not necessarily the same size. For example,

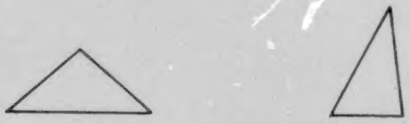
Any two circles are similar.



Any two squares are similar.

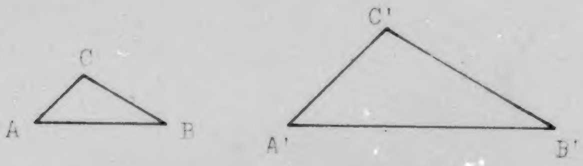
Any two line segments are similar.

Of course, it is not true that any two triangles are similar. For example, look at the two triangles below.



They certainly do not have the same shape.

On the other hand there are triangles that are similar. The two below are.

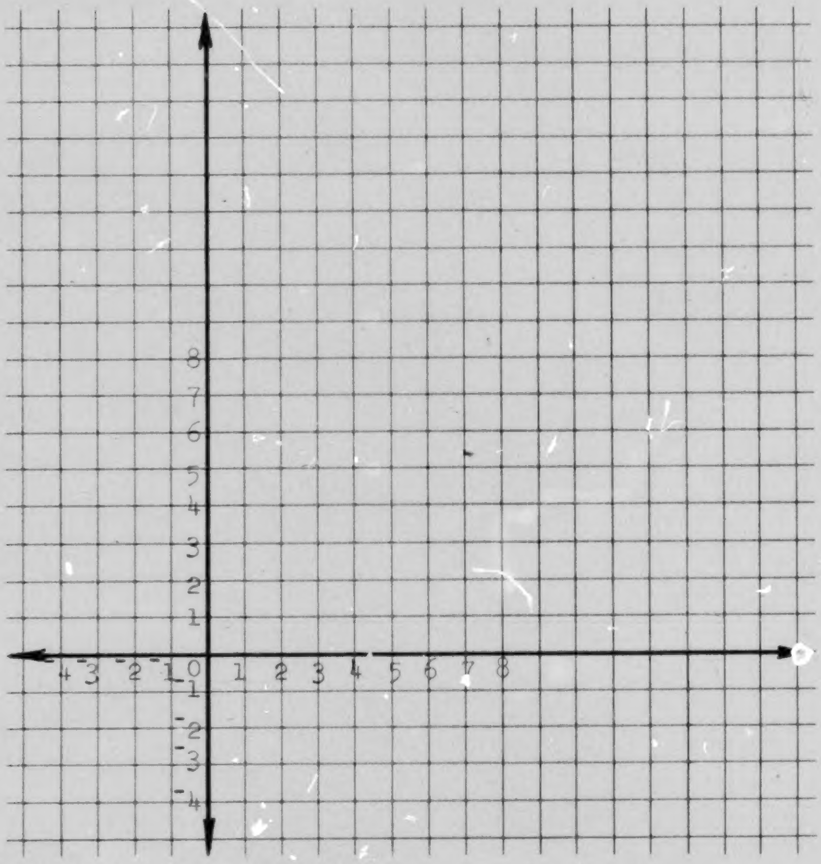


You can think about similar triangles in terms of shrinking or stretching. We can stretch the smaller triangle so that it is the size of the larger one. Or, we shrink the larger one down to the size of the smaller one.

The class discussion exercises that follow will show you how, mathematically, you can stretch or shrink a triangle to form a similar triangle.

Class Discussion

1. On the coordinate plane below plot and label the points A(0,0) , B(4,0) , and C(4,3) .



2. Draw segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$. (Use a straightedge.)

3. What kind of triangle has been formed? ________________

4. In $\triangle ABC$:

   (a) The length of side $\overline{AB}$ is ______ units.

   (b) The length of side $\overline{BC}$ is ______ units.

   (c) Is there any way we can count the number of units in the length of side $\overline{AC}$? ________

5. (a) Take your compass and place the needle point at the origin (point A) and the pencil point at point C .

   (b) Now draw an arc intersecting the x-axis.

   (c) What is the coordinate of the point where the arc you just drew intersects the x-axis? ____________

   (d) How many units in length does side $\overline{AC}$ seem to be?

   ____

6. The coordinates of points A , B , and C are written below. Multiply each coordinate for each point by 2 , thus finding the coordinates for points A', B', and C' .

A(0,0)	A'(__,__)
B(4,0)	B'(__,__)
C(4,3)	C'(__,__)

7. Plot and label the points B'(8,0) , and C'(8,6) on the same coordinate plane as you did for points A , B , and C . A'(0,0) is the same as A(0,0). We will refer to this point as A' when discussing $\triangle A'B'C'$.

8. Draw segments $\overline{A'B'}$, $\overline{B'C'}$, and $\overline{A'C'}$. (Use a straightedge.)

9. What kind of triangle has been formed? ________________

10. In $\triangle A'B'C'$

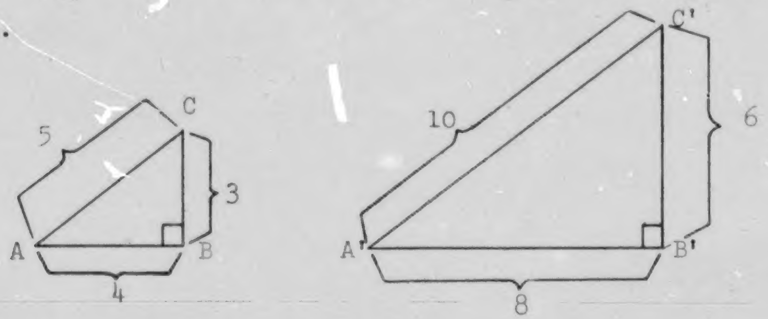
(a) The length of side $\overline{A'B'}$ is ______________ units.

(b) The length of side $\overline{B'C'}$ is ______________ units.

(c) How many units in length would you guess side $\overline{A'C'}$ to be? ______________

(d) Use the same process you used in Problem 5 and see if your guess is correct.

11. In order for you to see more clearly the two triangles you just drew, we have taken them "off the grid" and "separated" them.



Now let us compare the lengths of corresponding sides.

(a) The length of side $\overline{A'B'}$ is _____ times the length of side $\overline{AB}$.

(b) The length of side $\overline{B'C'}$ is _____ times the length of side $\overline{BC}$.

(c) The length of side $\overline{A'C'}$ is _____ times the length of side $\overline{AC}$.

You can see that the length of each side of the larger triangle is 2 times the length of the corresponding sides of the smaller triangle. In other words, we can stretch $\triangle ABC$ into $\triangle A'B'C'$ by multiplying the length of each side by 2 . In the same way we can shrink $\triangle A'B'C'$ into $\triangle ABC$ by multiplying the length of each side of $\triangle A'B'C'$ by $\frac{1}{2}$.

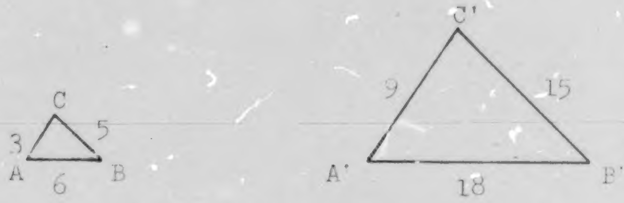
The numbers 2 and $\frac{1}{2}$ are called scale factors. In the triangles you drew, when we stretched the smaller triangle onto the larger triangle we multiplied by a scale factor of 2. If we go the other way and shrink the larger triangle onto the smaller triangle then we multiply by a scale factor of $\frac{1}{2}$.

You can see that when two figures are similar there are two scale factors, one for stretching and one for shrinking and they are the reciprocals of each other.

Exercises

1. In each of the problems below the triangles are similar. Find the scale factor for stretching and the scale factor for shrinking each triangle onto the other.

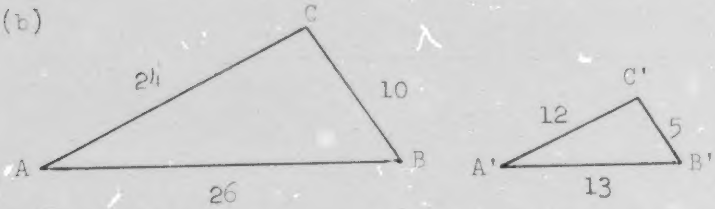
(a)



The scale factor for shrinking the larger triangle onto the smaller triangle is __________.

The scale factor for stretching the smaller triangle onto the larger triangle is __________.

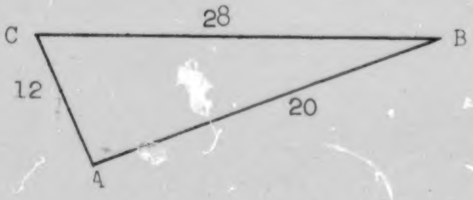
(b)

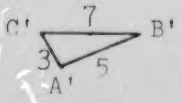


The scale factor for stretching the smaller triangle onto the larger triangle is __________.

The scale factor for shrinking the larger triangle onto the smaller triangle is __________.

(c)



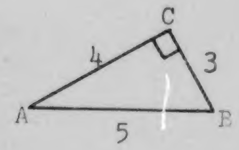


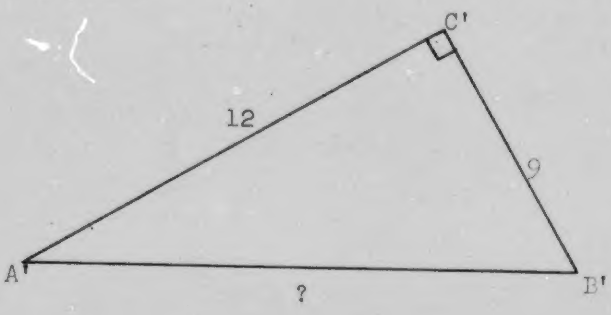
The scale factor for shrinking the larger triangle onto the smaller triangle is ___________.

The scale factor for stretching the smaller triangle onto the larger triangle is ___________.

2. In each problem below the triangles are similar. Decide whether the side whose length is not known is in the smaller or larger triangle. Then find the scale factor and length of the side of the triangle not given for you.

Example.

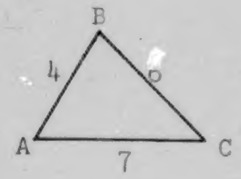


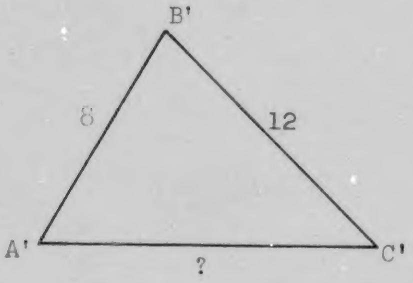


(i) The scale factor is 3 .

(ii) The length of side $\overline{A'B'}$ is 3 · 5 or 15 .

(a)

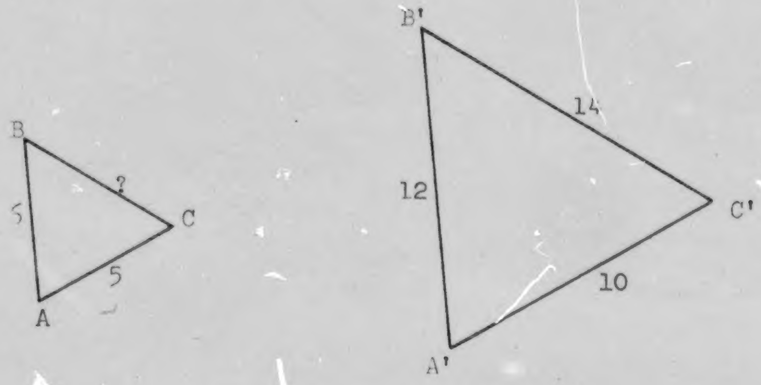




(i) The scale factor is _____.

(ii) The length of side $\overline{A'C'}$ is ____ · ____ or _____.

(b)


B
5
?
C
5
A
B'
14
12
C'
10
A'

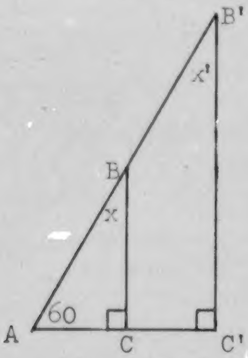
(i) The scale factor is _____.

(ii) The length of side $\overline{BC}$ is ___·___ or _____.

Similar Triangles

Class Discussion

Pictured below are two similar triangles,



much like the ones you drew in the last lesson.

1. Look at triangle ACB .
   What is the measure of ∠ ACB ? ____________

2. What is the sum of the measures of the angles of a triangle?
   ____

3. If the measure of the angle at A is 60 then what must be the measure of ∠ x ? ____________

4. Look at triangle AC'B'. What is the measure of ∠ AC'B' ?
   ____

5. What must be the measure of ∠ x' ? ____________

6. Fill in the blanks for the measures of each angle for △ ABC and △ AB'C'.

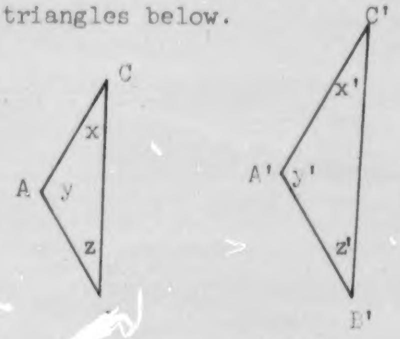
	△ ABC	△ AB'C'
(a)	m ∠ A = _____	m ∠ A = _____
(b)	m ∠ ACB = _____	m ∠ AC'B' = _____
(c)	m ∠ x = _____	m ∠ x' = _____

(d) Complete the sentence below.

If two triangles are similar the corresponding angles have ________ measure.

Before we go on, let us review exactly what we mean by corresponding sides and corresponding angles.

Look at the two similar triangles below.



$\overline{AB}$ and $\overline{A'B'}$ are corresponding sides

$\overline{AC}$ and $\overline{A'C'}$ are corresponding sides

$\overline{CB}$ and $\overline{C'B'}$ are corresponding sides

∠ x and ∠ x' are corresponding angles

∠ y and ∠ y' are corresponding angles

∠ z and ∠ z' are corresponding angles

Now we know how to tell if two triangles are similar:

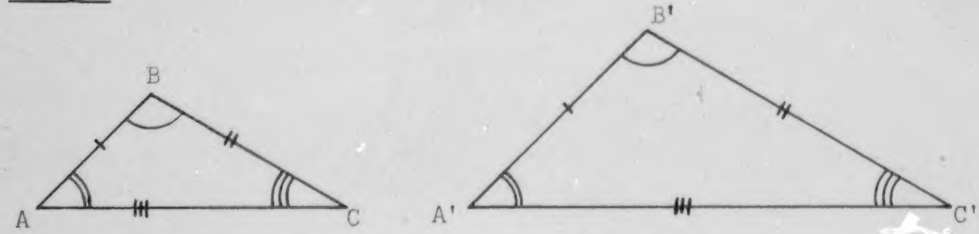
(1) Their corresponding angles must be equal in measure,

and (2) each pair of corresponding sides must have the same scale factor.

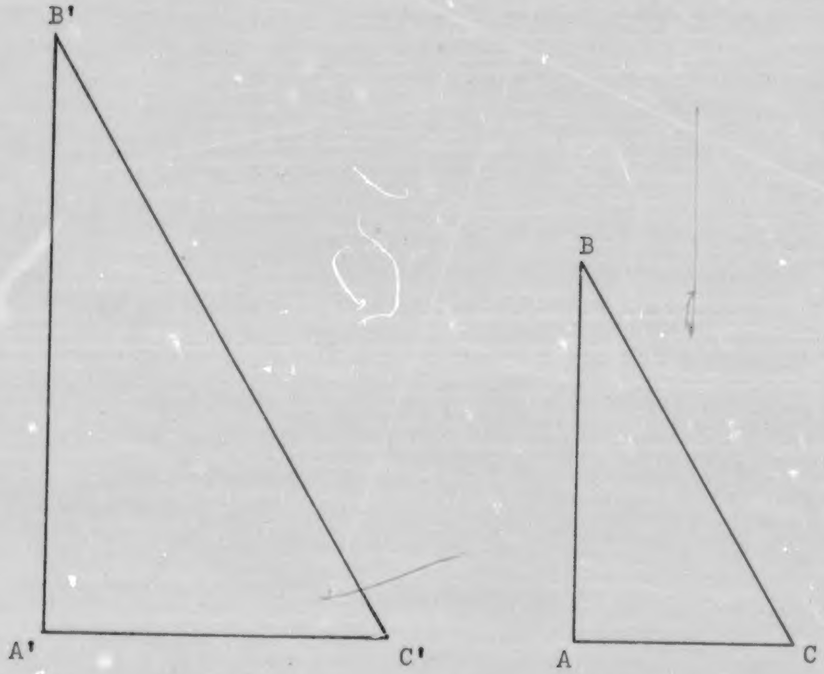
Exercises

The triangles in each pair are similar. Mark pairs of corresponding congruent angles and pairs of corresponding sides as is shown in the example.

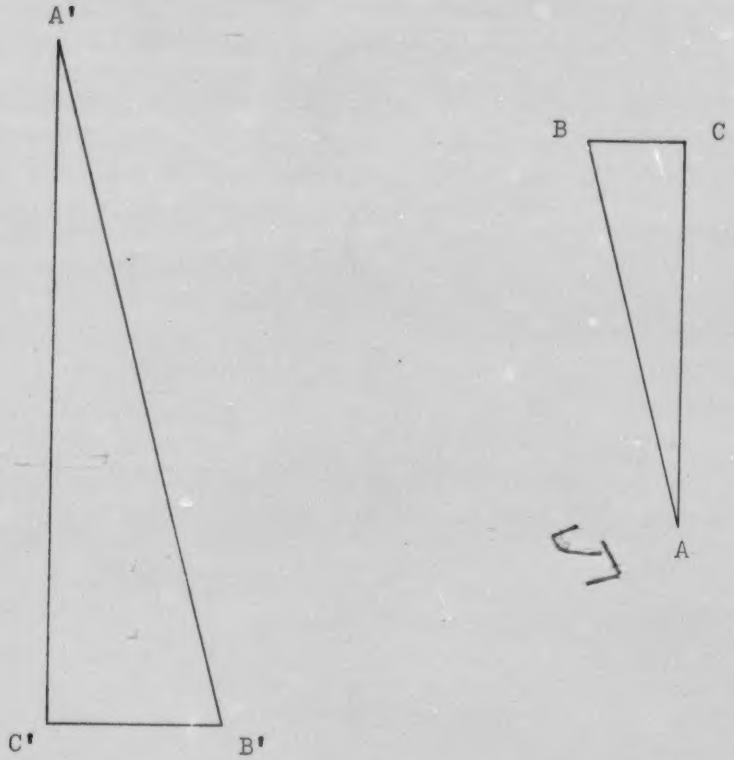
Example.



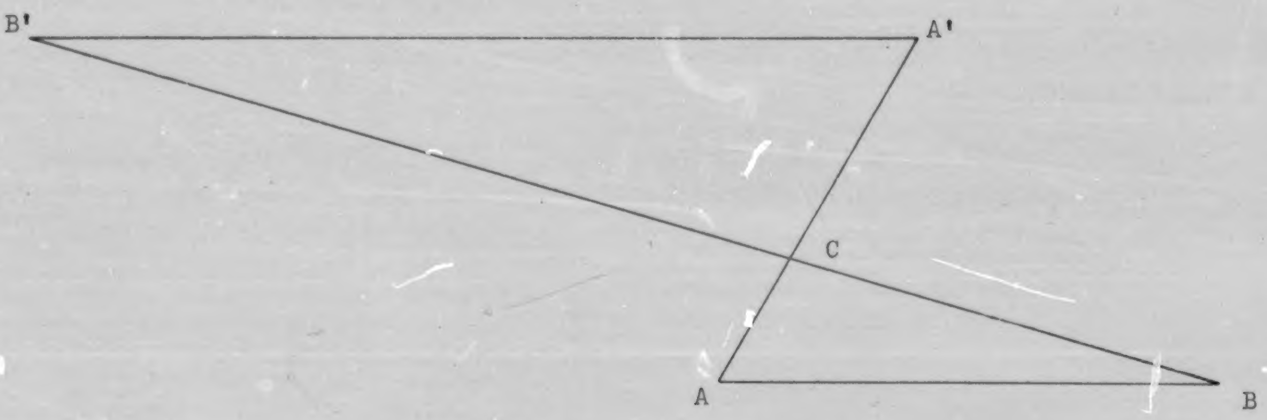
1.



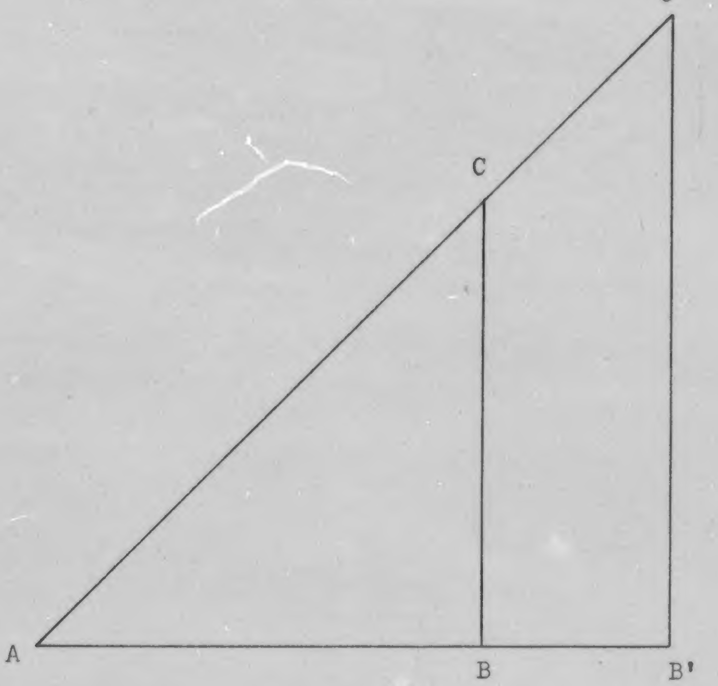
2.



3.

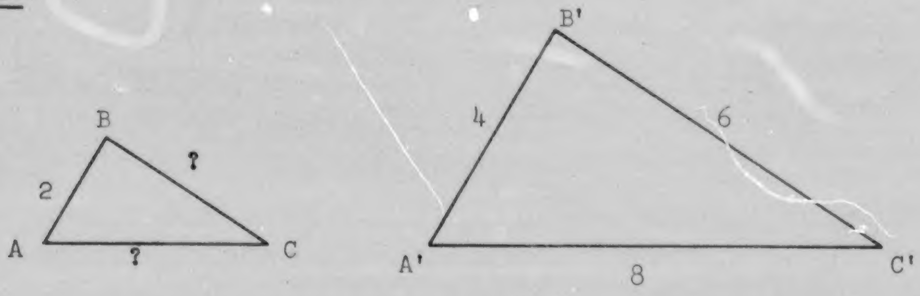


4.



5. For each pair of similar triangles find the scale factor. Then find the missing lengths of the remaining sides.

Example.

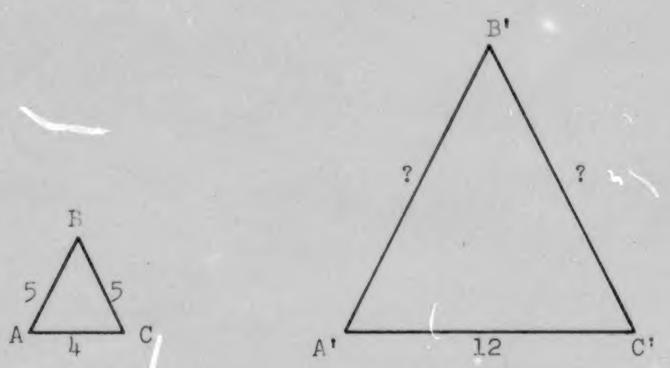


(i) The scale factor is $\underline{\frac{1}{2}}$.

(ii) The length of $\overline{BC}$ is $\underline{\frac{1}{2}}$ · $\underline{6}$ or $\underline{3}$.

(iii) The length of $\overline{AC}$ is $\underline{\frac{1}{2}}$ · $\underline{8}$ or $\underline{4}$.

(a)

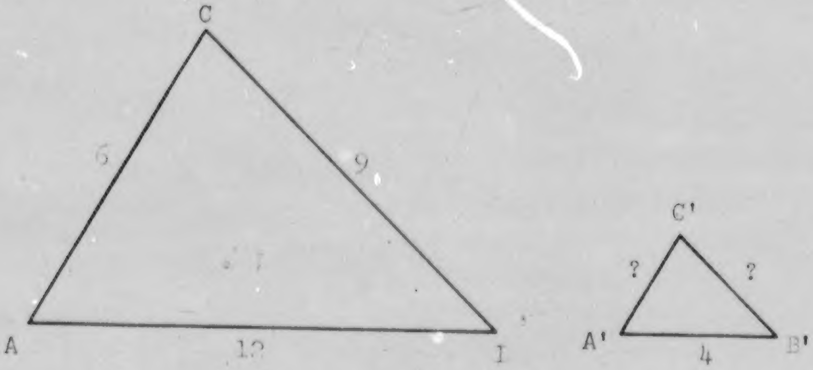


(i) The scale factor is _____ .

(ii) The length of $\overline{C'B'}$ is _____ · _____ or _____.

(iii) The length of $\overline{A'B'}$ is _____ · _____ or _____.

(b)



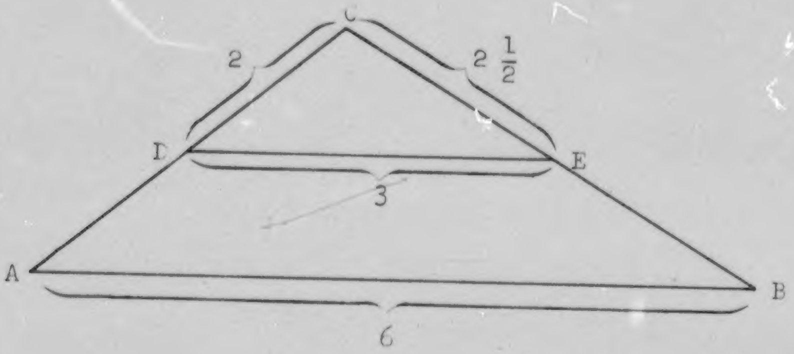
(i) The scale factor is ____.

(ii) The length of $\overline{A'C'}$ is __ · __ or ____.

(iii) The length of $\overline{B'C'}$ is __ · __ or ____.

BRAINBOOSTER.

6. Find the scale factor and the missing lengths in the similar triangles below.



(i) The scale factor is ____.

(ii) The length of side $\overline{AC}$ is __ · __ or ____.

(iii) The length of side $\overline{BC}$ is __ · __ or ____.

Ratios and Scale Factors

In an earlier chapter you learned that the ratio of the number "a" to the number "b" is written as the fraction

$\frac{a}{b}$ (of course, "b" cannot be zero).

As examples, the ratio of:

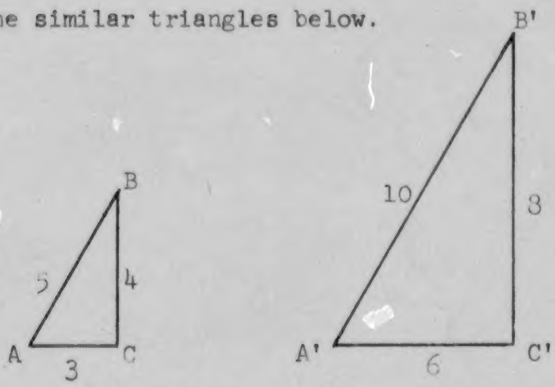
3 to 4 is $\frac{3}{4}$,

5 to 7 is $\frac{5}{7}$,

7 to 10 is $\frac{7}{10}$,

8 to 3 is $\frac{8}{3}$.

Look at the similar triangles below.



You can see that to stretch $\triangle ABC$ onto $\triangle A'B'C'$ we would use a scale factor of 2. Or, to shrink $\triangle A'B'C'$ onto $\triangle ABC$ we would use a scale factor of $\frac{1}{2}$.

An easy way to find the scale factor for two similar triangles is to compare the lengths of corresponding sides by writing them as a ratio. In the drawings above:

$$\frac{\text{length of } \overline{A'B'}}{\text{length of } \overline{AB}} = \frac{10}{5} = 2$$

$$\frac{\text{length of } \overline{A'C'}}{\text{length of } \overline{AC}} = \frac{6}{3} = 2$$

$$\frac{\text{length of } \overline{B'C'}}{\text{length of } \overline{BC}} = \frac{8}{4} = 2$$

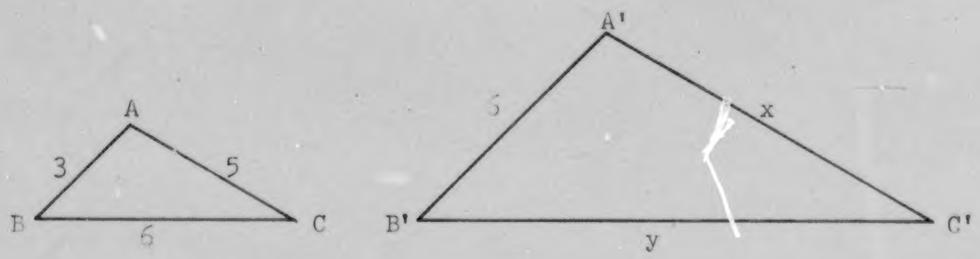
Now this tells us that

$$\frac{\text{length of } \overline{A'B'}}{\text{length of } \overline{AB}} = \frac{\text{length of } \overline{A'C'}}{\text{length of } \overline{AC}} = \frac{\text{length of } \overline{B'C'}}{\text{length of } \overline{BC}},$$

that is, the ratios of the lengths of corresponding sides of similar triangles are equal.

We can now use this idea of equal ratios to solve problems involving similar triangles.

Class Discussion



The two triangles above are similar. We want to find the lengths of sides $\overline{A'C'}$ and $\overline{B'C'}$.

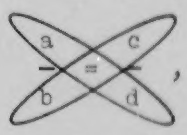
1. The ratio $\dfrac{\text{length of } \overline{A'B'}}{\text{length of } \overline{AB}} = \dfrac{6}{\_\_\_\_\_\_}$

2. The ratio $\dfrac{\text{length of } \overline{A'C'}}{\text{length of } \overline{AC}} = \dfrac{x}{\_\_\_\_\_\_}$

3. Knowing that the ratios of the lengths of corresponding sides of similar triangles are equal gives

$\dfrac{x}{\_\_\_\_} = \dfrac{6}{\_\_\_\_}$ (Write in the missing denominators.)

4. You know that if $\frac{a}{b} = \frac{c}{d}$, then by multiplying as shown,



, $a \cdot d = b \cdot c$.

This is how we compare two rational numbers and is called the Comparison Property. Use the Comparison Property and fill in the blanks.

$$\underline{\quad} \cdot x = \underline{\quad} \cdot 6$$

$$\text{so,} \quad 3 \cdot x = 30$$

5. To solve the equation $3 \cdot x = 30$ we must multiply both sides of the equation by $\frac{1}{3}$. So $\underline{\quad} \cdot 3 \cdot x = \underline{\quad} \cdot 30$ and $x = \underline{\quad}$.

6. The value for x, and thus the length of side $\overline{A'C'}$, is $\underline{\qquad\qquad}$.

Let us use the same procedure to find the length of side $\overline{B'C'}$.

7. The ratio $\frac{\text{length of } \overline{A'B'}}{\text{length of } \overline{AB}} = \frac{\underline{\quad}}{\underline{\qquad\quad}}$

8. The ratio $\frac{\text{length of } \overline{B'C'}}{\text{length of } \overline{BC}} = \frac{y}{\underline{\qquad\quad}}$

9. Knowing that the ratios of the lengths of corresponding sides of similar triangles are equal gives

$$\frac{y}{\underline{\quad}} = \frac{6}{\underline{\quad}} \quad \text{(Write in the missing denominators.)}$$

10. Use the comparison property and fill in the missing numbers.

$$\underline{\qquad} \cdot y = \underline{\qquad} \cdot \underline{\qquad}$$

$$\text{so,} \quad 3 \cdot y = 36$$

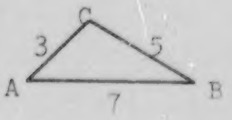
11. To solve the equation $3 \cdot y = 36$ we multiply both sides of the equation by $\underline{\qquad}$. So, $\underline{\quad} \cdot 3 \cdot y = \underline{\quad} \cdot 36$ and $y = \underline{\quad}$.

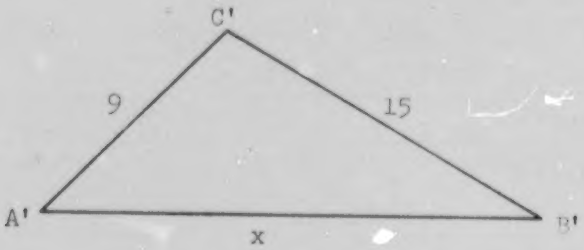
12. The value for y, and thus the length of side $\overline{B'C'}$ is $\underline{\qquad}$.

Exercises

Use equal ratios to find the missing lengths. The triangles in each problem are similar.

1.

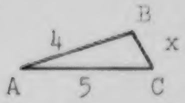


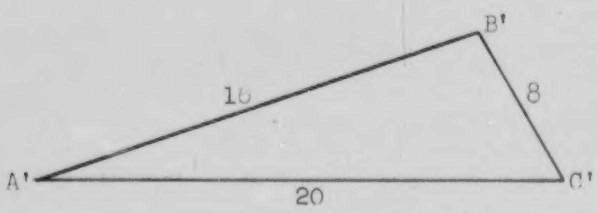


(Show your work in the space below.)

x = ________

2.

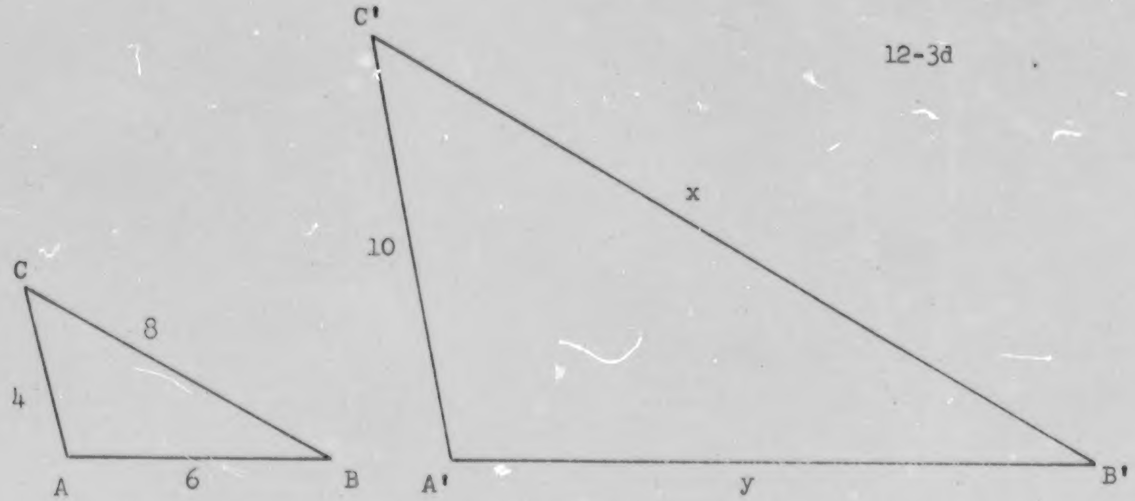




(Show your work in the space below.)

x = ________

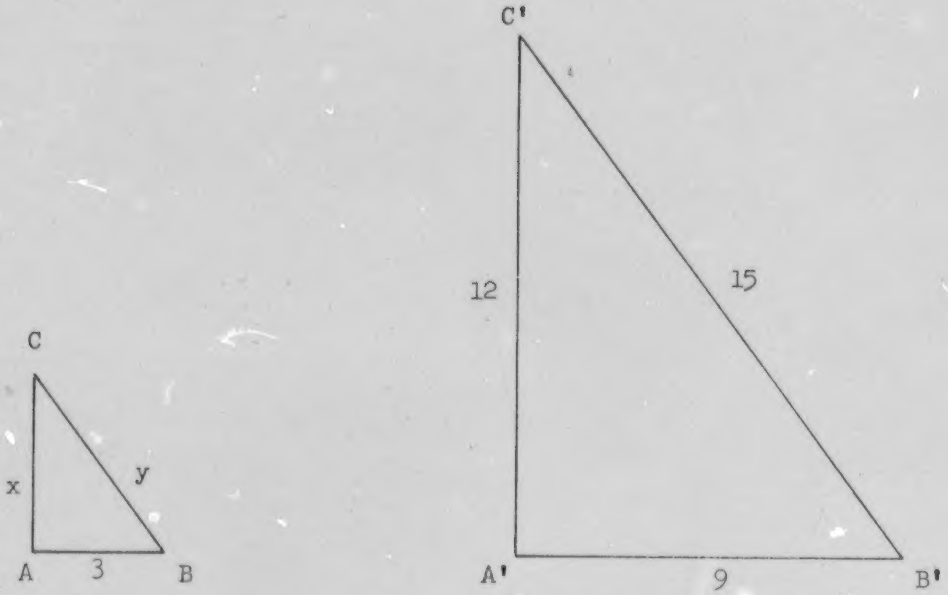
3.


C'
x
10
C
8
4
A
6
B
A'
y
B'

(Show your work in the space below.)

x = ________ y = ________

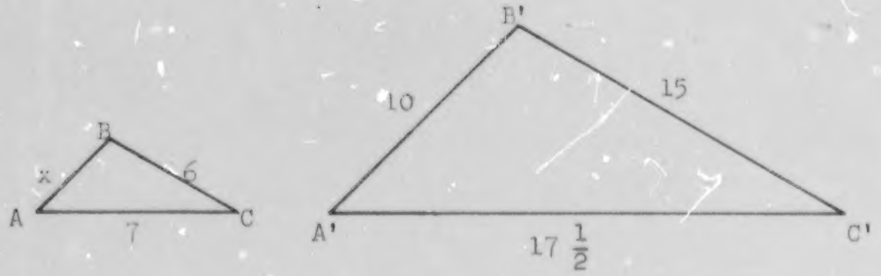
4.


C'
12
15
C
y
x
A
3
B
A'
9
B'

(Show your work in the space below.)

x = ________ y = ________

5.


B
x
6
A
7
C
B'
10
15
A'
17 1/2
C'

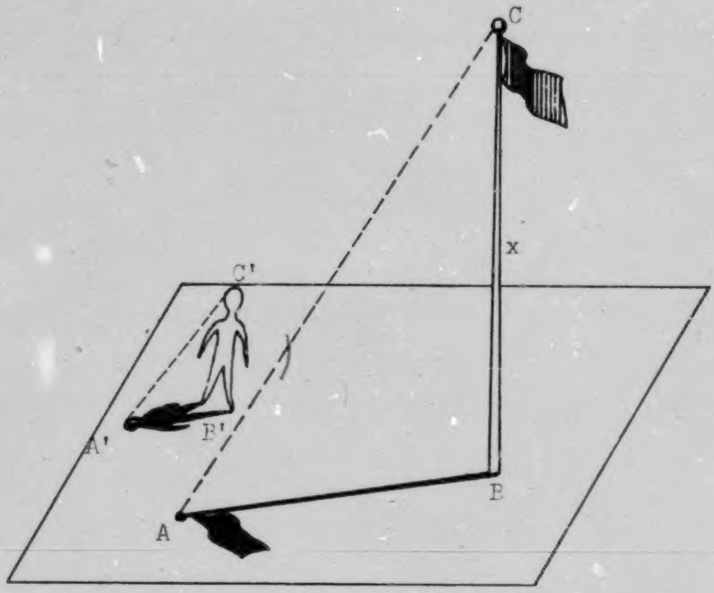
(Show your work in the space below.)

x = __________

6. In Problem 3, what is the scale factor for stretching $\triangle ABC$ onto $\triangle A'B'C'$? ____________________

BRAINBOOSTER.

7. We have a flagpole which we want to find the height of. We do not want to cut it down nor do we want to climb it. How do we find the height of the flagpole?



Directions:

(i) Measure the length of the flagpole's shadow, $\overline{AB}$.
(ii) Measure the length of a friend's shadow, $\overline{A'B'}$, at the same time.
(iii) Measure the friend's height $\overline{B'C'}$.

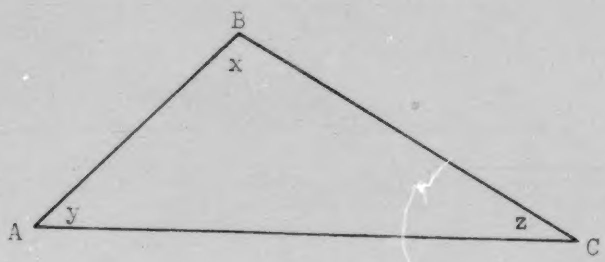
If the two triangles are similar (and they are) you can easily solve this problem since

$$\frac{\text{length of the pole's shadow}}{\text{length of the friend's shadow}} = \frac{\text{height of the pole } (x)}{\text{height of the friend}}$$

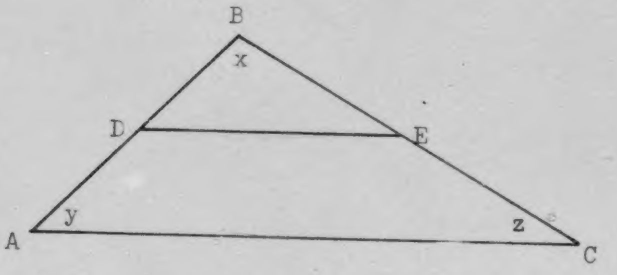
If the friend is 6 feet tall and casts a 4 foot shadow, then how high is the pole if its shadow is 24 feet? ____________

How a Photo Enlarger Works; Parallels and Similarity

If we have a triangle, such as pictured below



and we draw a segment $\overline{DE}$, parallel to one of the sides.



then the triangle formed, $\triangle DEB$, is similar to the triangle we started with, $\triangle ACB$.

A photo enlarger is a fairly simple application of this idea. Basically, it is just a box with a horizontal glass shelf in the middle. The box is light-proof except for a pinhole in the top through which light may pass. (In order to admit more light, lenses are used instead of a pinhole, but the light coming through the lenses acts as though it came from a pinhole.)

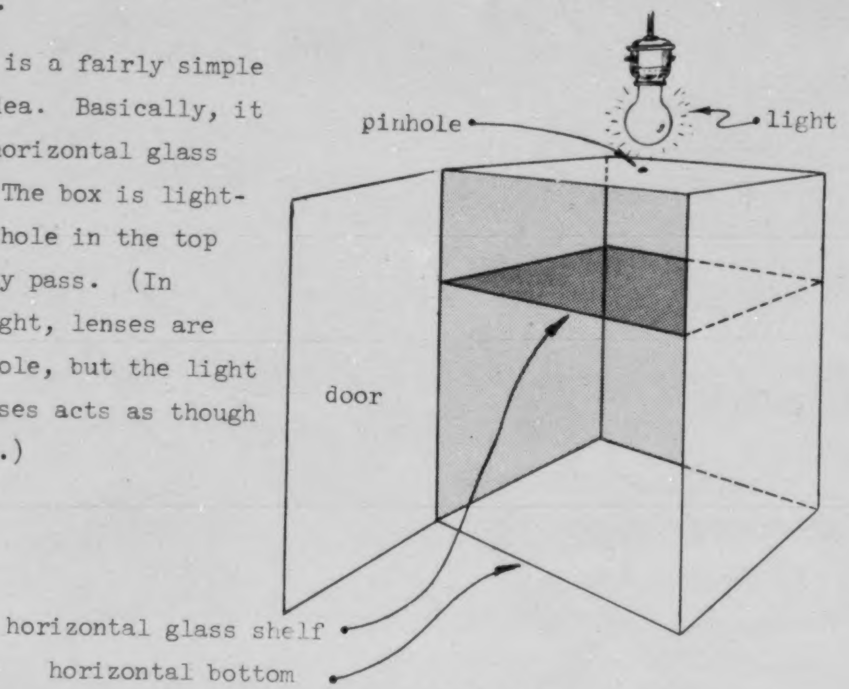
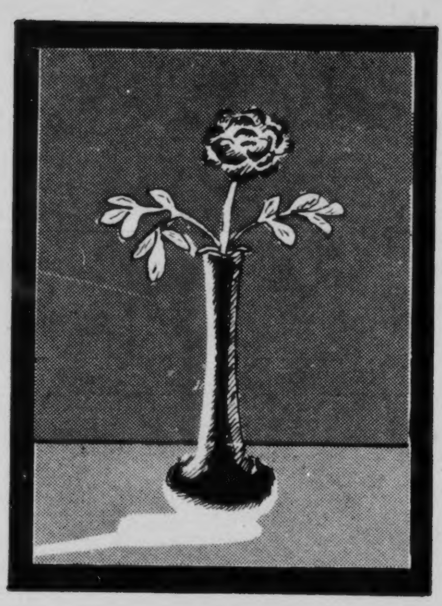

pinhole
light
door
horizontal glass shelf
horizontal bottom

Figure 1

In the dark, a photo negative is placed on the glass shelf and a piece of light-sensitive photo paper is placed on the bottom. The door is then closed and the light is turned on. At any point on the photo negative, the shading determines how much light can pass through to reach a point on the paper. Thus a correspondence is established between points on the photo negative and points on the paper.



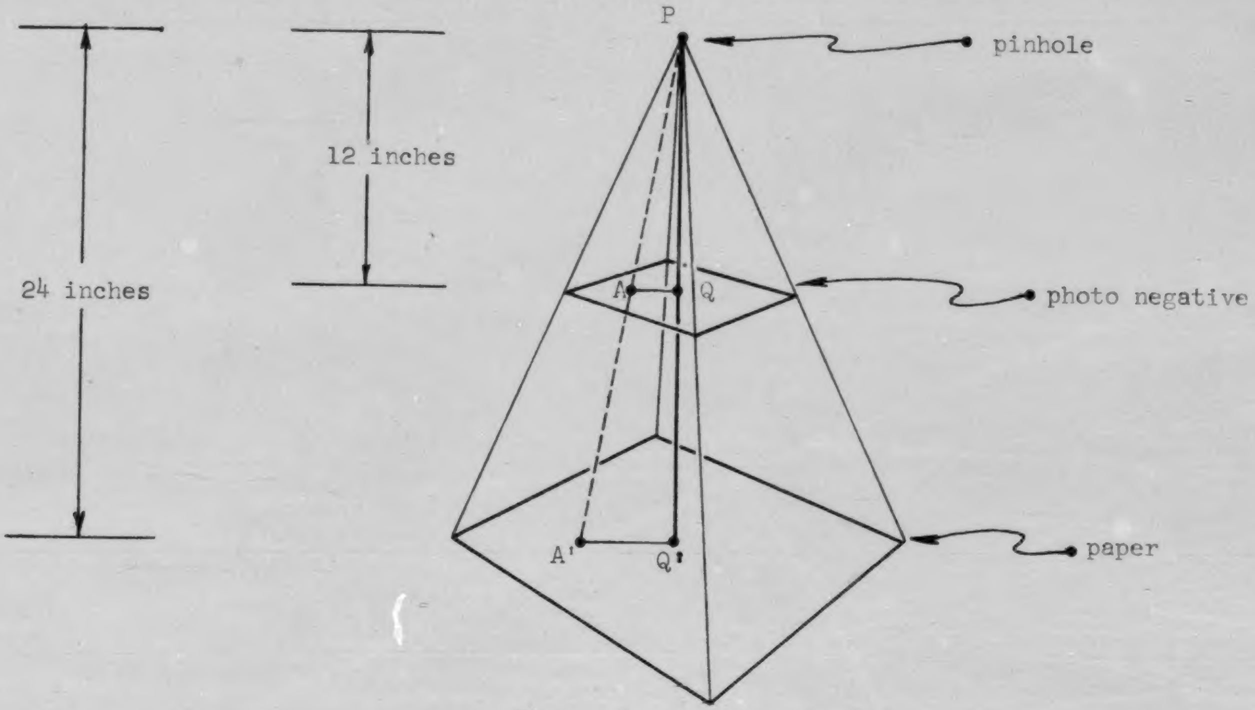


Figure 2

We will say that the vertical distances from the pinhole to the two planes are 12 inches and 24 inches.

You can see that $\overline{QA}$ is parallel to $\overline{Q'A'}$, so that

$$\triangle PQA \text{ is similar to } \triangle PQ'A' .$$

The scale factor in stretching $\triangle PQA$ onto $\triangle PQ'A'$ is

$$\frac{\text{length of } \overline{PQ'}}{\text{length of } \overline{PQ}} = \frac{24}{12} = 2 .$$

Now if we take any other point B on the photo negative (as in Figure 3) then $\overline{A'B'}$ is parallel to $\overline{AB}$,

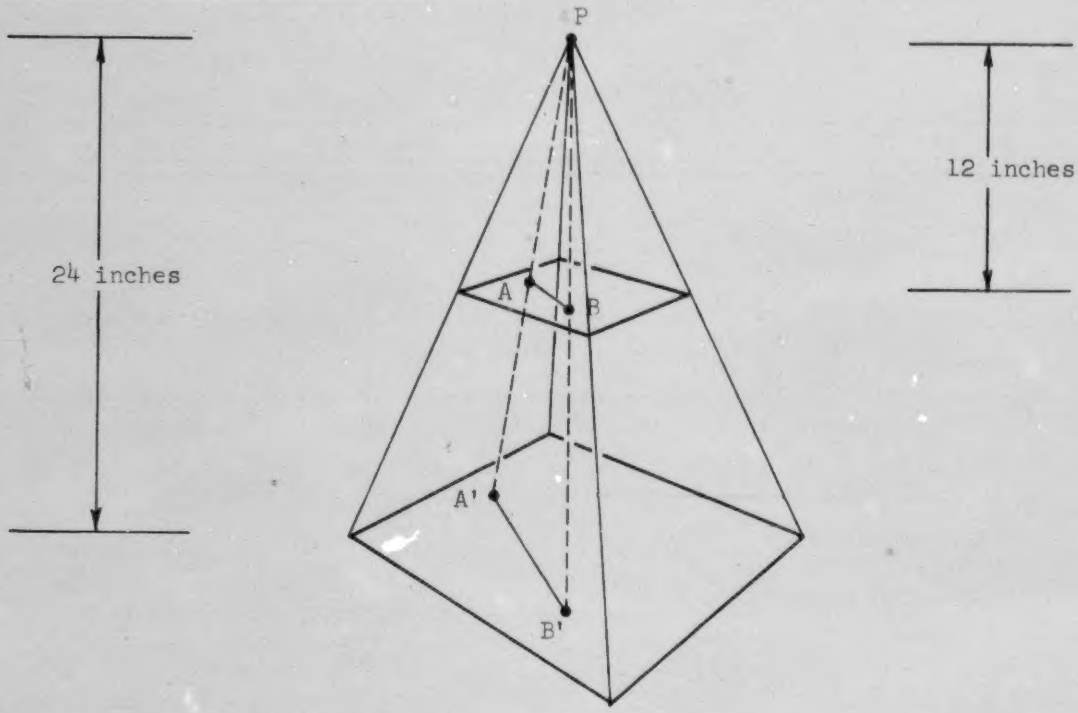


Figure 3

and we have $\triangle PBA$ similar to $\triangle PB'A'$. Again the scale factor in stretching $\triangle PBA$ onto $\triangle PB'A'$ is

$$\frac{\text{length of } \overline{PA'}}{\text{length of } \overline{PA}} = \frac{24}{12} \text{ or } 2 .$$

You can see that the distance between any two points in the enlargement is always twice the corresponding distance on the photo negative.

Class Discussion

Suppose we use the same photo enlarger and want an enlargement that will be three times as large as the photo negative. How far from the top should the shelf be placed in order to do this?

Solution.

1. In Figure 3, $\triangle PBA$ is similar to $\triangle PB'A'$.

2. Then

$$\frac{\text{length of } \overline{PA'}}{\text{length of } \overline{PA}} = \text{scale factor}$$

is the ratio that will give us a scale factor.

3. Since the length of $\overline{PA'}$ is 24 and the desired scale factor is 3, we can then write

$$\frac{24}{(\text{length of } \overline{PA})} = \frac{3}{1}$$

4. Using the comparison property

$$1 \cdot 24 = 3 \cdot (\text{length of } \overline{PA})$$

5. Multiplying both sides of the equation by $\frac{1}{3}$

$$(\tfrac{1}{3}) \cdot 24 = (\tfrac{1}{3}) \cdot 3 \cdot (\text{length of } \overline{PA})$$

$$8 = (\text{length of } \overline{PA})$$

Thus, the shelf with the photo negative on it should be 8 inches from the top.

Exercises

The glass shelf in the photo enlarger is movable. How far from the top should the shelf be placed in order to get enlargements with the following scale factors? Use the fact that

$$\frac{24}{\text{length of } \overline{PA}} = \frac{\text{scale factor}}{1}$$

1. A scale factor of 4 .

Distance from top = ________ inches.

2. A scale factor of 6 .

Distance from top = ________ inches.

3. A scale factor of 12 .

Distance from top = ________ inches.

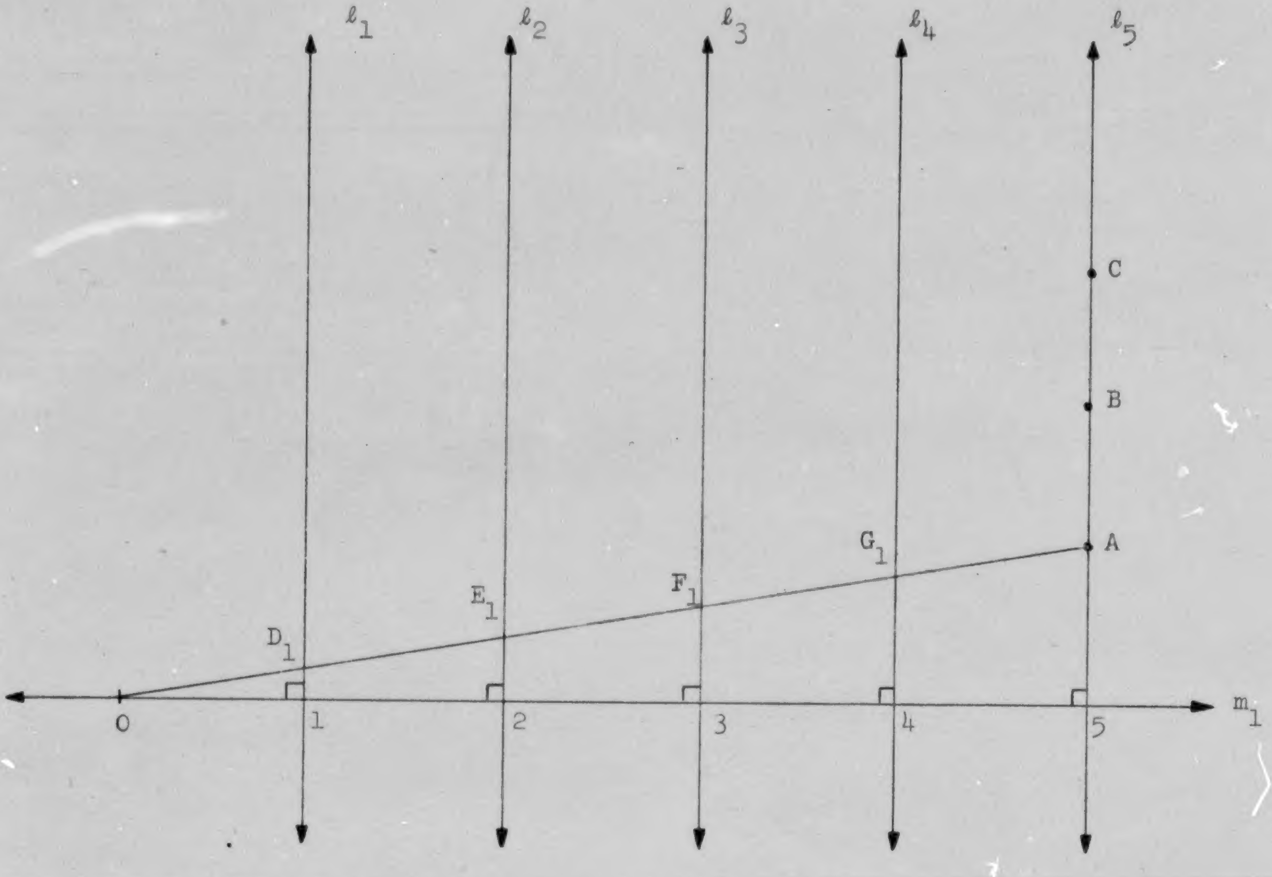
BRAINBOOSTER.

4. A scale factor of 5 .

Distance from top = ________ inches.

How to Divide Up a Line Segment

Pictured below is a number line with perpendicular lines drawn through the points 1 through 5 .



Because ℓ_1 , ℓ_2 , ℓ_3 , ℓ_4 , and ℓ_5 are all perpendicular to the same line, you know that they are parallel to each other. Furthermore, as they pass through the points 1 , 2 , 3 , 4 , and 5 which are equally spaced on the number line, you know that these parallel lines are equally spaced.

We have drawn segment $\overline{OA}$ which intersects ℓ_1 , ℓ_2 , ℓ_3 , and ℓ_4 at points D_1 , E_1 , F_1 , and G_1 .

Class Discussion

1. Take your compass and place the needle point at O and the pencil point at D_1 .
2. Without changing your compass setting, place the needle point at D_1 and the pencil point at E_1 .
3. Is $\overline{OD_1} \cong \overline{D_1E_1}$? ______________
4. Place the needle point at E_1 and the pencil point at F_1 .

   Is $\overline{D_1E_1} \cong \overline{E_1F_1}$? ______________
5. Again, without changing your setting, use your compass to find out if $\overline{OD_1}$, $\overline{D_1E_1}$, $\overline{E_1F_1}$, $\overline{F_1G_1}$, and $\overline{G_1A}$ are all congruent to each other.

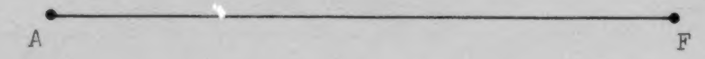
   Are they? ___________
6. Use a straightedge to draw segment $\overline{OB}$.
7. Use your compass to find out whether the segments cut off by the parallel lines in $\overline{OB}$ are all congruent to each other.

   Are they? ___________
8. Use a straightedge to draw $\overline{OC}$.
9. Use your compass to find out whether the segments cut off by the parallel lines in $\overline{OC}$ are all congruent to each other.

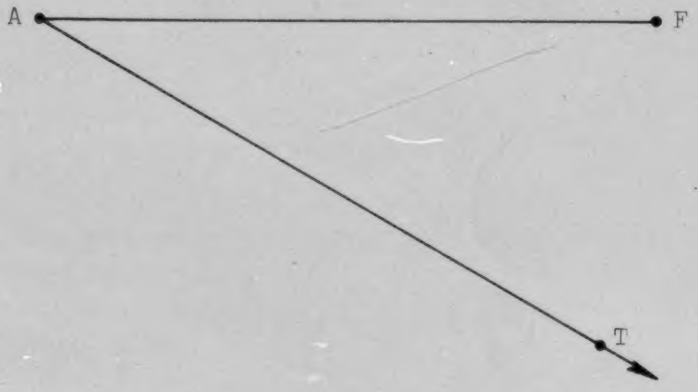
   Are they? ___________
10. Do you agree that, no matter what line segment is drawn so that it passes through these equally spaced parallel lines, the segments they cut off will all be congruent to each other? ___________

Now we will show you how to use these ideas to divide a line segment into smaller congruent line segments.

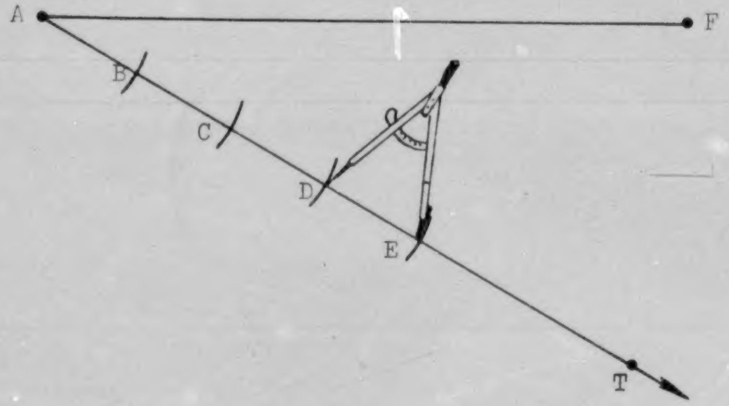
Suppose we want to divide $\overline{AF}$ into four smaller congruent line segments.



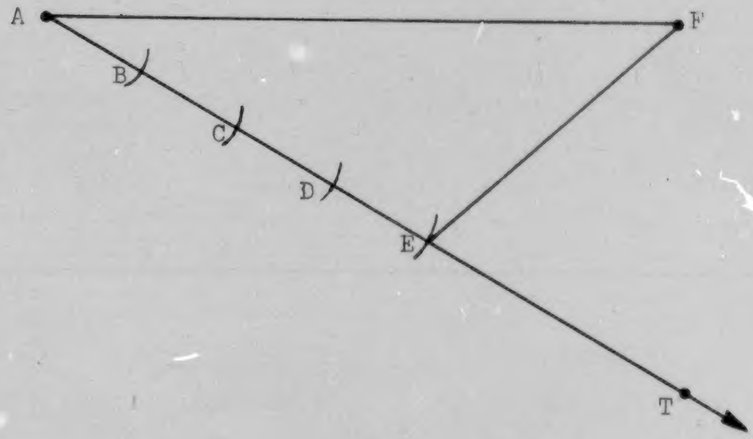
Draw a ray $\overrightarrow{AT}$.



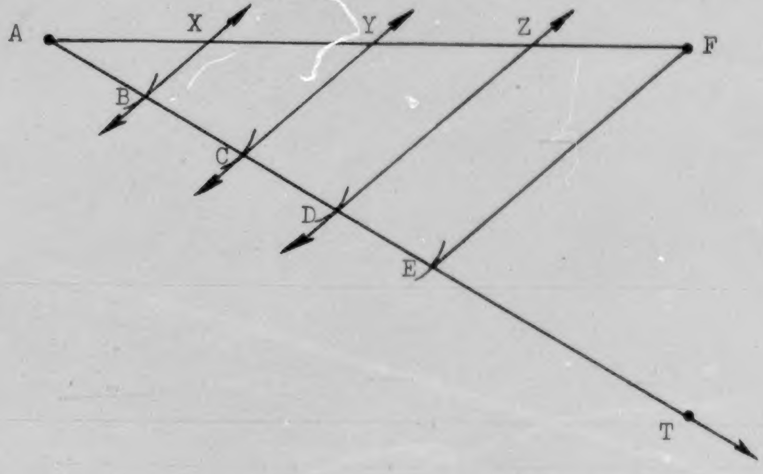
We take a compass and lay off four congruent segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DE}$ on ray $\overrightarrow{AT}$.



Next, we draw $\overline{EF}$.



Now, if we construct the lines through points B , C , and D parallel to $\overline{EF}$, they will intersect $\overline{AF}$ in three points, x , y , and z ,

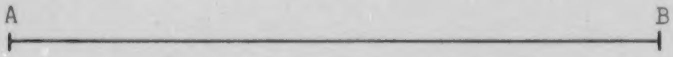


and $\overline{AX}$, $\overline{XY}$, $\overline{YZ}$, and $\overline{ZF}$ will be congruent to each other. Thus, we have divided $\overline{AF}$ into four smaller congruent segments. We have also formed four triangles -- $\triangle$ AEF and three smaller triangles all similar to $\triangle$ AEF .

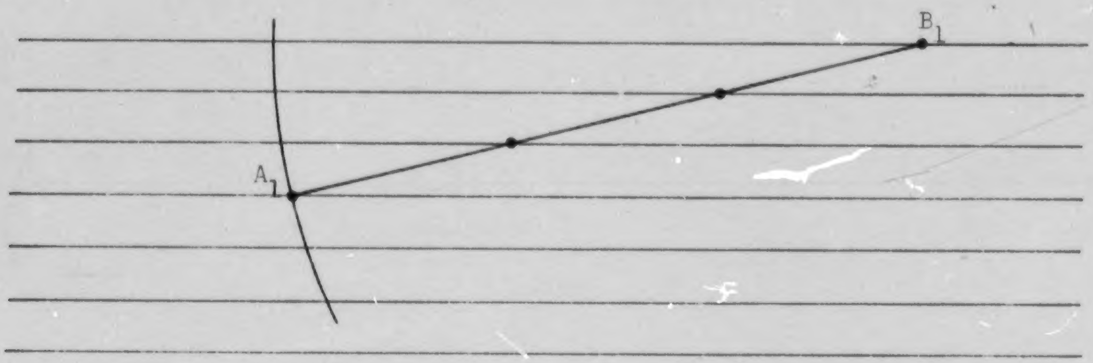
The problem with using the above method is that constructing lines that are parallel is not an easy task. Fortunately, you always have equally spaced parallel lines available to you. For example, the lines on your notebook paper are parallel and equally spaced.

We will use the equally spaced parallel lines on a piece of notebook paper to divide a line segment into smaller congruent line segments.

Problem. Divide line segment $\overline{AB}$ into three smaller congruent line segments.



The equally spaced parallel lines below are much like those in a piece of notebook paper.

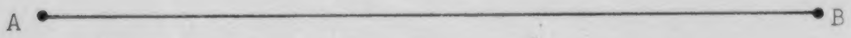


1. First, we pick a point B_1 on one of the top lines.
2. We place the needle point of the compass on point B and the pencil point on point A .
3. Without changing the compass setting we place the needle point on B_1 and draw an arc intersecting the lines. Any line segment drawn from B_1 to a point on the arc.will be congruent to $\overline{AB}$.
4. We want to divide $\overline{AB}$ into three smaller congruent line segments so we find the point where the arc intersects the third line down from the line that contains B_1 and label that point A_1 .
5. Now we draw $\overline{A_1B_1}$. Now the parallel lines clearly divide segment $\overline{A_1B_1}$ in three smaller congruent segments. As $\overline{A_1B_1} \cong \overline{AB}$ all you need to do is take your compass and lay off these congruent segments on $\overline{AB}$.

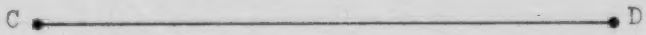
Exercises

Use the equally spaced parallel lines below to divide the given line segments into smaller congruent segments.

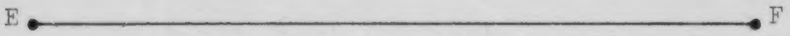
1. Divide $\overline{AB}$ into three smaller congruent line segments.



2. Divide $\overline{CD}$ into five smaller congruent line segments.

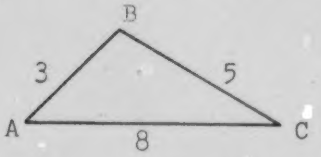


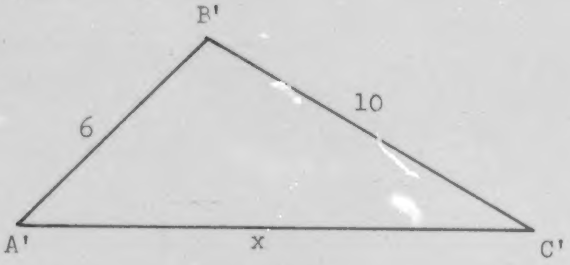
3. Divide $\overline{EF}$ into seven smaller congruent line segments.



Ratios and Similar Triangles

Earlier in this chapter you found that the ratios of corresponding sides of similar triangles are equal. Knowing this, you were able to solve problems involving similar triangles. For example, to find the missing length of $\overline{A'C'}$ in the two similar triangles below.





You first wrote

$$\frac{\text{length of } \overline{A'B'}}{\text{length of } \overline{AB}} = \frac{6}{3}$$

then

$$\frac{\text{length of } \overline{A'C'}}{\text{length of } \overline{AC}} = \frac{x}{8}$$

You know that these two ratios are equal so

$$\frac{x}{8} = \frac{6}{3}$$

By the comparison property

$$3 \cdot x = 8 \cdot 6$$

or

$$3 \cdot x = 48$$

Multiplying both sides of the equation by $\frac{1}{3}$, you get

$$\left(\frac{1}{3}\right) \cdot 3 \cdot x = \frac{1}{3}(48)$$

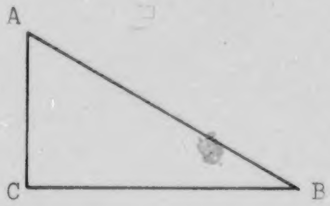
or

$$x = 16$$

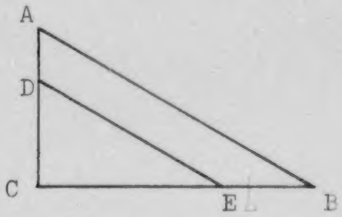
Thus, the length of $\overline{A'C'}$ is 16 . Of course, if you saw that the scale factor was 2 then all you had to do was multiply $2 \cdot 8$ and you would have had the answer. Often, though, it is not easy to see what the scale factor is.

Class Discussion

In the last lesson you learned that if you have a triangle,

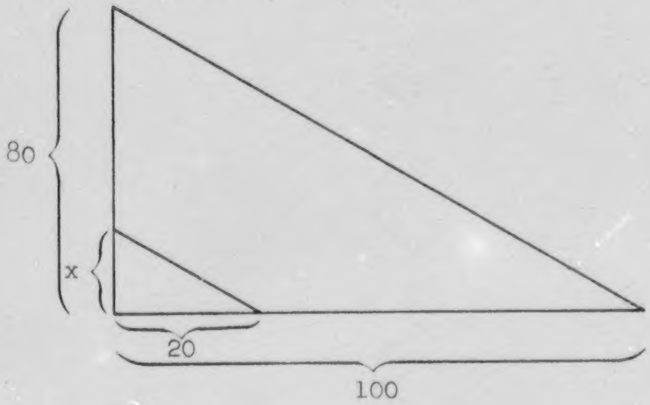


and you drew a line segment, $\overline{DE}$, parallel to one of the sides



then the triangle formed, $\triangle DCE$, is similar to $\triangle ACB$.

Suppose you have two similar triangles and the lengths of certain sides of the triangles are given, as is shown below.



Can we find the length of the side x ? Your knowledge of equal ratios and the comparison property will let you find the length of x .

1. Suppose we want the scale factor for shrinking the larger triangle onto the smaller triangle. Then, using the sides of the triangles that are horizontal, the scale factor, written as a ratio, would be __________.

2. If we use the sides of the triangle that are vertical, then the same scale factor, written as a ratio, would be __________.

3. Now we know the scale factors are equal and therefore the ratios are equal, so

$$\underline{\quad x \quad} = \underline{\qquad} .$$

4. Using the comparison property we get

$$\underline{\quad} \cdot x = \underline{\quad} \cdot \underline{\quad}$$

$$\text{So,} \quad 100 \cdot x = 1600$$

5. To solve this equation we multiply both sides by __________.

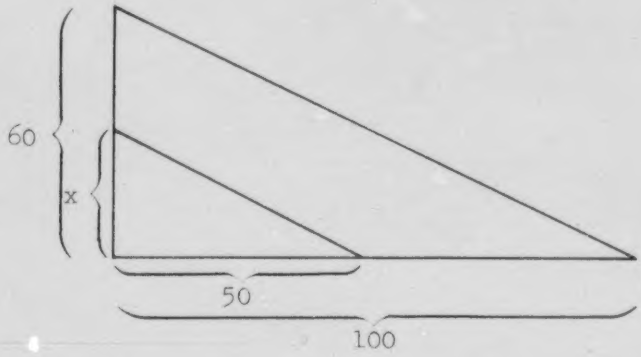
6. After doing the arithmetic we find that

$$x = \underline{\qquad} .$$

Exercises

The triangles in each problem are similar. Use equal ratios to find a value for x.

1 .



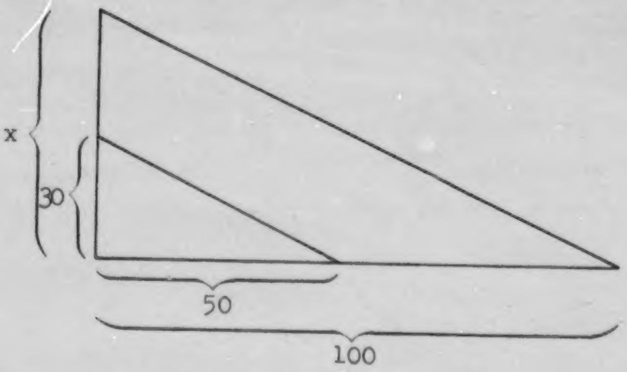
(a) Write the equal ratios here. ______________________________

(b) Use the comparison property. ______________________________

(c) Multiply both sides of the equation by the correct number, which is ______________________________.

(d) Do the arithmetic to get the answer. $x =$ __________

2.



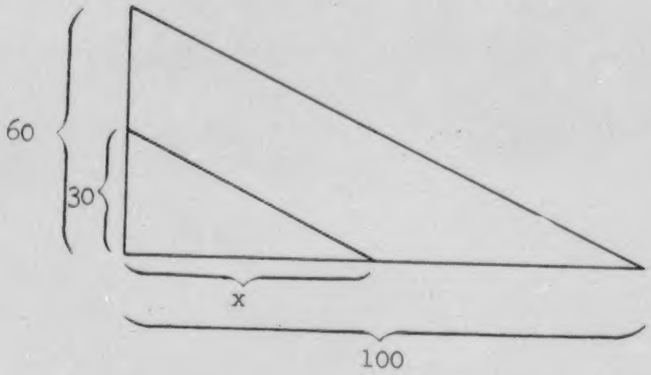
(a) Write the equal ratios here. ________________

(b) Use the comparison property. ________________

(c) Multiply both of the equations by the correct number, which is ________________.

(d) Do the arithmetic to get the answer. x = ______

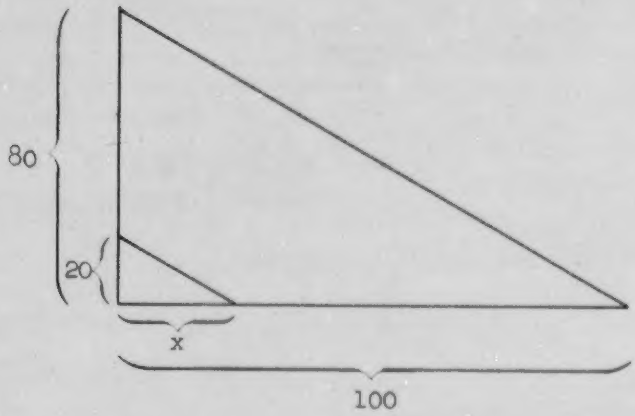
3.



(Do your work below.)

x = ________

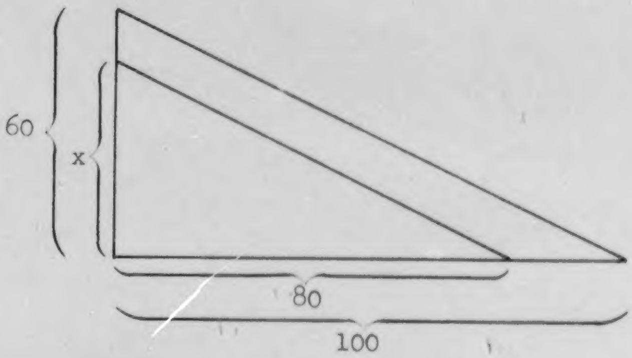
4.



(Do your work below.)

x = ____________

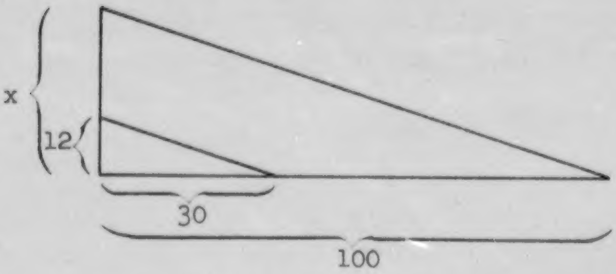
5.



(Do your work below.)

x = ____________

6.



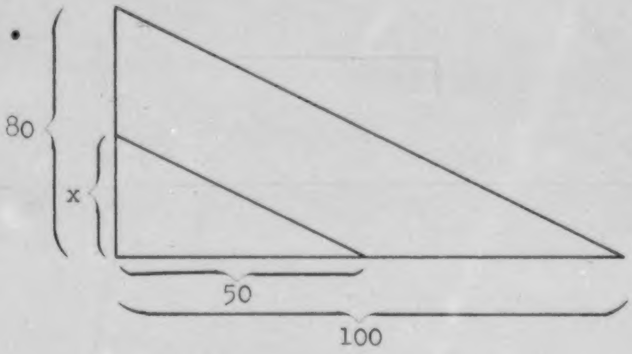
(Do your work below.)

x = ________

Similar Triangles, Ratios, and Percent

In the last lesson you may have noticed that in every problem the larger triangle of the two similar triangles had a side of length 100.

In shrinking the larger triangle onto the smaller one you wrote a ratio in which the 100 was in the denominator. For example:



Using the horizontal sides of the two triangles you wrote

$$\frac{50}{100},$$

and then using the vertical sides you wrote the ratio

$$\frac{x}{80}.$$

As these two ratios represent the shrinking scale factor and as scale factors in similar triangles are equal, you then could write

$$\frac{x}{80} = \frac{50}{100}.$$

Your knowledge of the comparison property and your ability to solve equations then let you do as follows:

(a) $\frac{x}{80} = \frac{50}{100}$

(b) $100 \cdot x = 50 \cdot 80$

(c) $100 \cdot x = 4000$

(d) $(\frac{1}{100}) \cdot 100 \cdot x = (\frac{1}{100}) \cdot 4000$

(e) $x = \frac{4000}{100}$

(f) $x = 40$

Although you may not have realized it, when you worked these problems you were also solving a percent problem.

Whenever a ratio is written with 100 in the denominator you are writing that ratio as a percent. For example,

$\frac{50}{100}$ may be read as 50 percent.

As the symbol for percent is %, we can then write

$$\frac{50}{100} = 50 \cdot \frac{1}{100} = 50\% .$$

You can see that $\frac{1}{100}$ is 1% .

Class Discussion

Let us now use the idea of similar triangles to help us solve a percent problem.

Problem. 30% of 80 is what number?

Picture Solution:

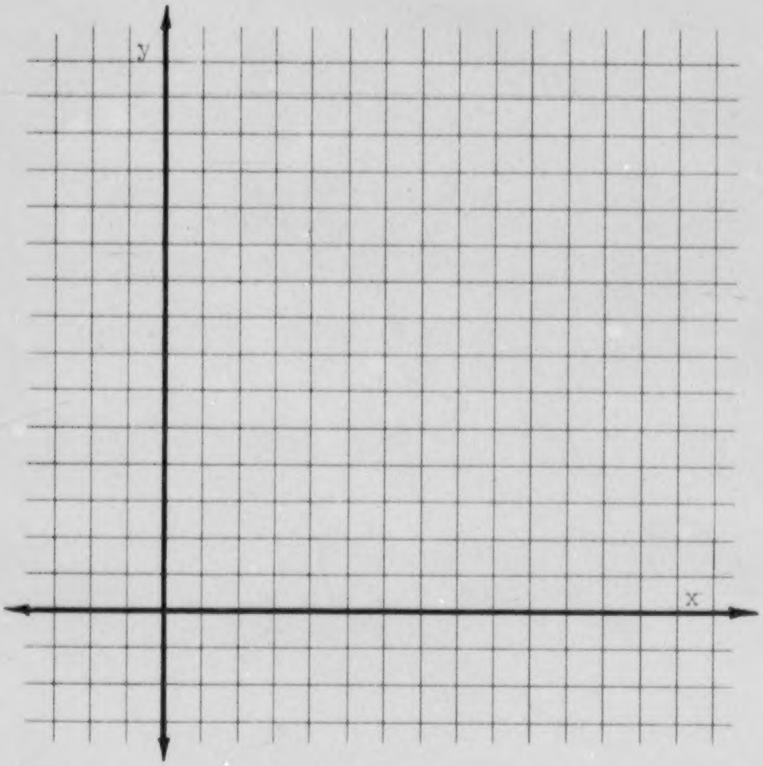
(a) On the grid on the next page, count over 10 squares on the x-axis and mark it 100 . This will represent the horizontal leg of the larger triangle.

(b) Now count over 3 squares on the x-axis and mark it 30 . This will represent the horizontal leg of the smaller triangle. Notice that you now have pictured the ratio $\frac{30}{100}$ or 30% .

(c) Count up 8 squares on the y-axis and mark that point 80 . Use a straightedge to draw the segment that connects the point marked 100 to the point marked 80 . You now have drawn the larger of the two similar triangles.

(d) Now, put your straightedge on the point marked 30 and as nearly as you can draw a line parallel to the segment you drew from 100 to 80, so that it cuts the y-axis. Mark this point t. You now have drawn the smaller of the two similar triangles.

(e) Make a guess as to where the point t lies on the y-axis. ______________________



Arithmetic Solution:

$$\frac{30}{100} = \frac{t}{80}$$

$$30 \cdot 80 = 100 \cdot t$$

$$2400 = 100 \cdot t$$

$$\left(\frac{1}{100}\right) \cdot 2400 = \left(\frac{1}{100}\right) \cdot 100 \cdot t$$

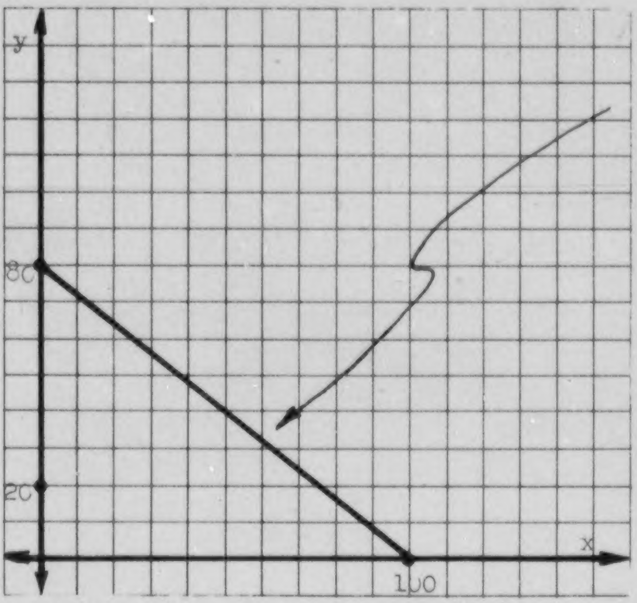
$$24 = t$$

Therefore, 30% of 80 is 24. How close was your guess to this answer?

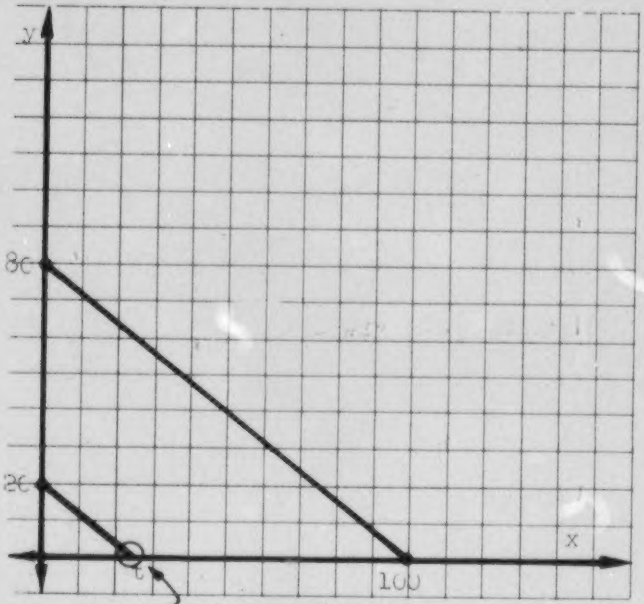
Exercises

Each of the problems in this exercise set is a percent problem pictured on a grid. We have drawn one of the two similar triangles for you. You are to draw the other triangle by drawing a line through the marked point parallel to the given line and then estimate the answer.

Example. What % of 80 is 20 ?



You are given this type of picture.

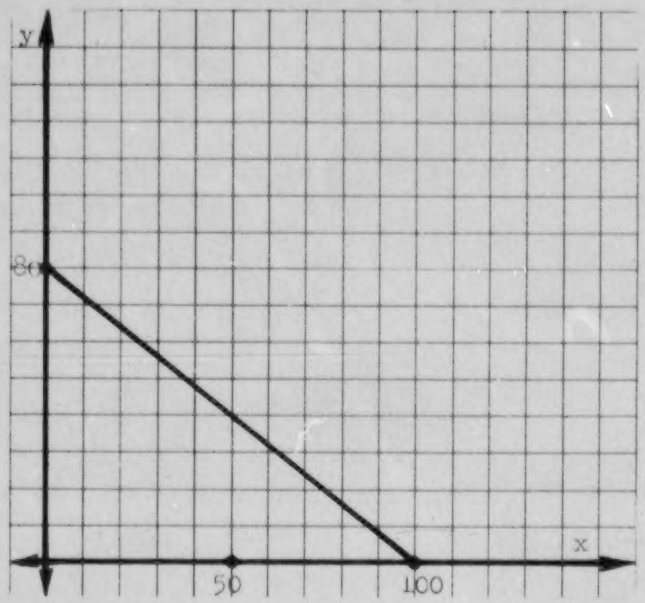


You complete the picture like this.

What % of 80 is 20 ?
About 25 % .

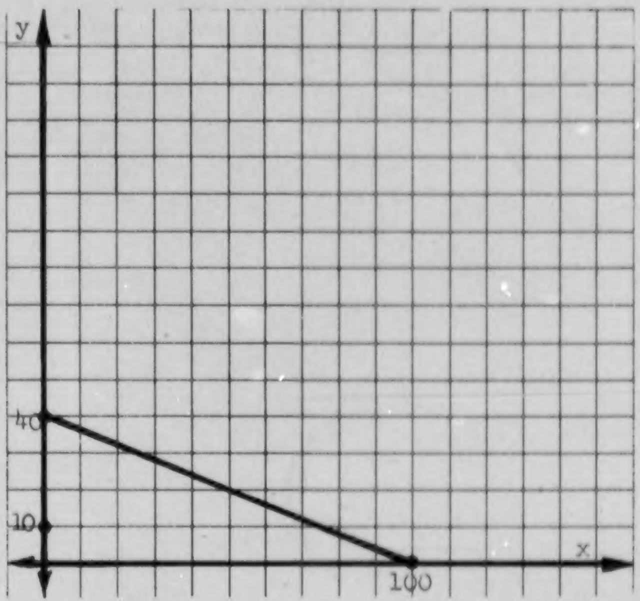
Estimate the value of this point.

1. What is 50% of 80 ?



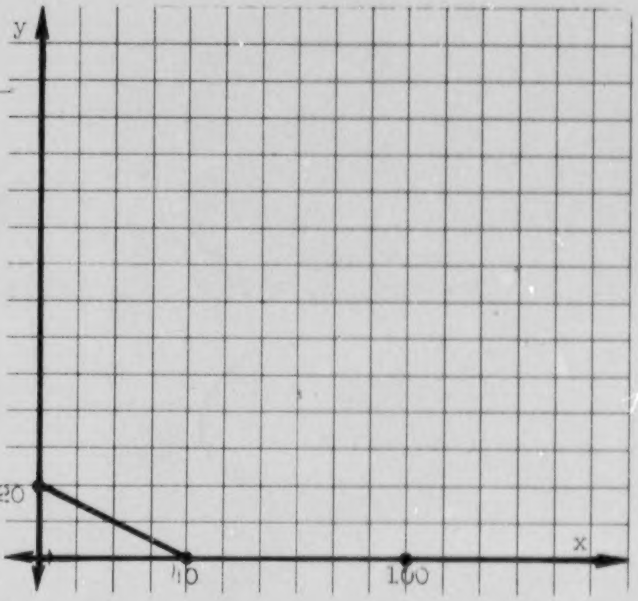
50% of 80 is about ________.

2. What % of 40 is 10 ?



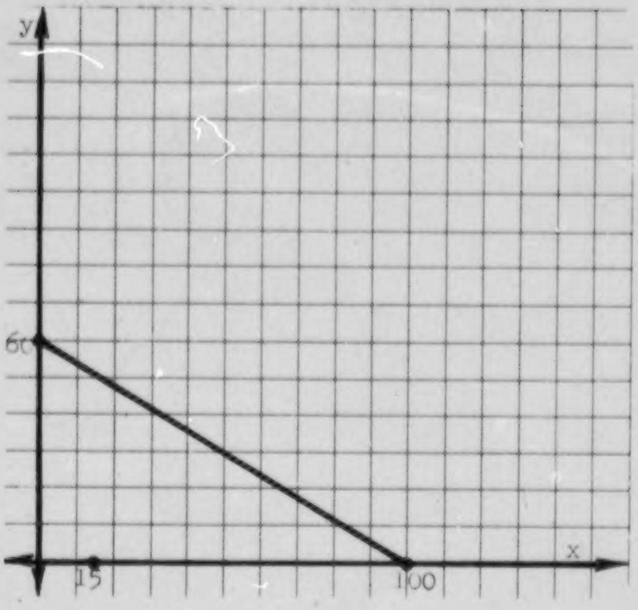
10 is about ______% of 40 ?

3. 40% of what number is 20 ?



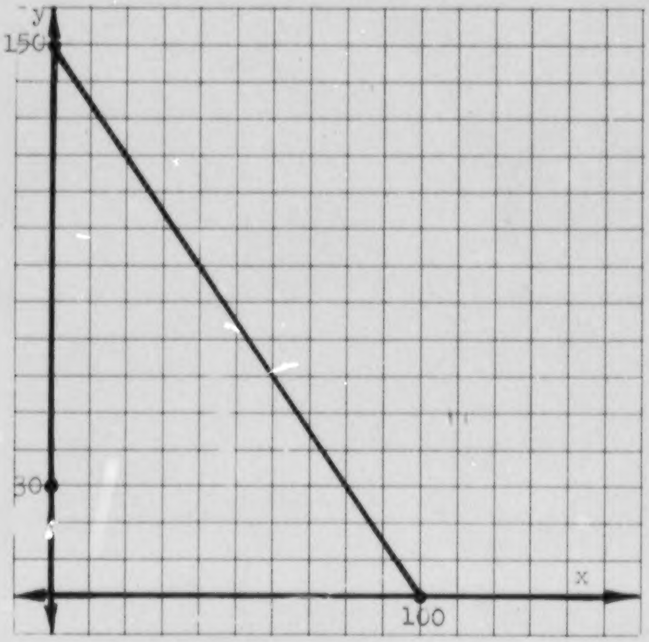
40% of about ______ is 20 .

4. 15% of 60 is what number?



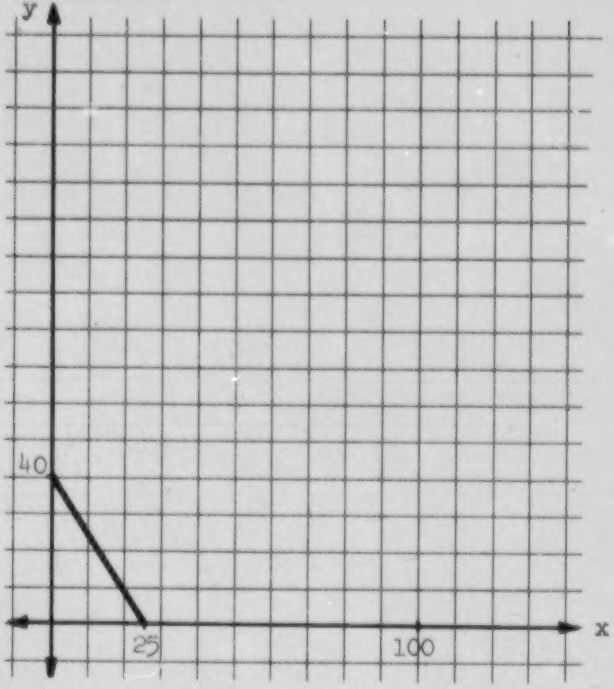
15% of 60 is about ________.

5. What % of 150 is 30 ?



30 is about ______% of 150 .

6. 25% of what number is 40 ?



25% of about ________ is 40 .

Here are the solutions to each of the problems you just finished doing. If your estimates are within 5 of these solutions you have done well.

1. 50% of 80 is 40 .

2. 10 is 25% of 40 .

3. 40% of 50 is 20 .

4. 15% of 60 is 9 .

5. 30 is 20% of 150 .

6. 25% of 160 is 40 .

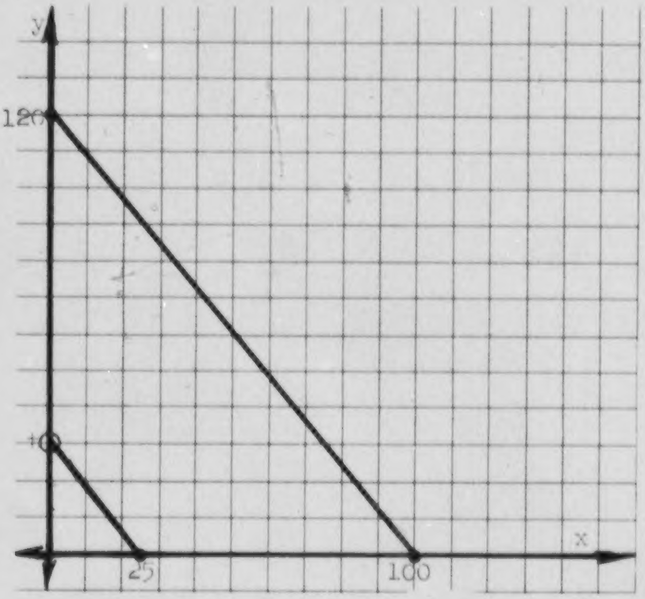
Solving Percent Problems

Suppose you are given the problem,

"25% of 120 is what number?".

You know that you can picture this problem on a grid and estimate the answer. In most cases your estimate will be close. But we need to be more than close. We need to be able to get an exact answer. Picturing percent problems on a grid not only lets you estimate the answer but it also lets you see the ratios so that you can get an exact answer.

Let us picture the problem, "25% of 120 is what number?", on a grid and see if we can arrive at the exact solution, not just an estimate.



Looking at our picture we can estimate that the answer should be close to 30, but we are really not sure.

We do know that the ratios of corresponding sides of similar triangles are equal, so we can write:

$$\frac{t}{120} = \frac{25}{100} .$$

Using our knowledge of the comparison property and our ability to solve equations, we then write:

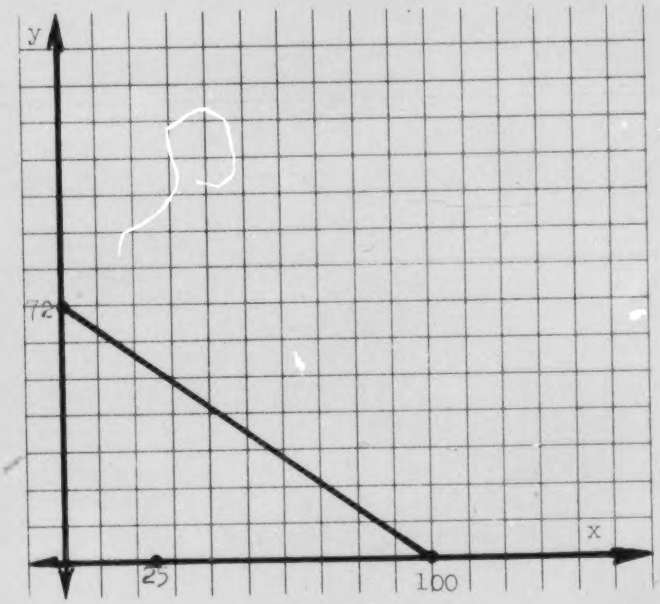
$$\begin{aligned} 100 \cdot t &= 25 \cdot 120 , \\ 100 \cdot t &= 3000 , \\ \left(\frac{1}{100}\right) \cdot 100 \cdot t &= \left(\frac{1}{100}\right) \cdot 3000 , \\ t &= 30 . \end{aligned}$$

We now have found the solution to the problem and we see that our estimate was a good one.

Exercises

For each problem, make an estimate of the solution by drawing similar triangles on the grid. Then use the comparison property and your ability to solve equations to find the arithmetic solution.

1. 25% of 72 = ?



(a) Write the equal ratios here. ______________________________

(b) Use the comparison property. ______________________________

(c) Multiply both sides of the equation by the correct number.

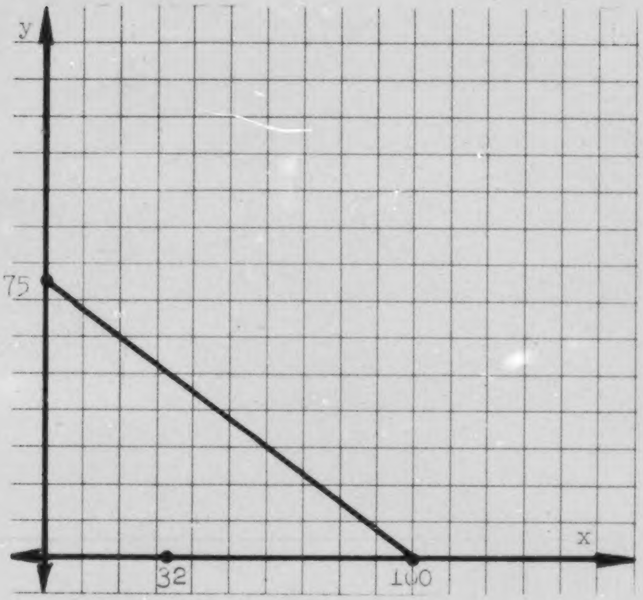
____

(d) Do the arithmetic to get the answer.

25% of 72 = __________.

2. 32% of 75 = ?

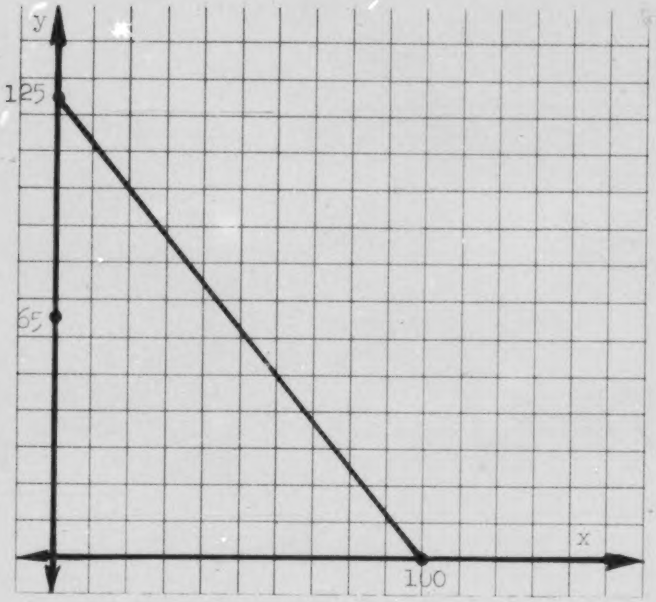
(Show arithmetic solution below.)



32% of 75 = ________.

3. What % of 125 is 65 ?

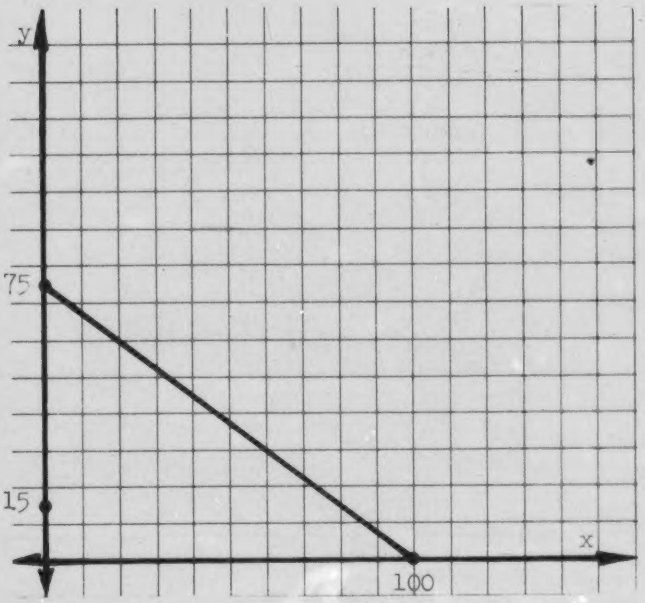
(Show arithmetic solution below.)



________% of 125 is 65 .

4. What % of 75 is 15 ?

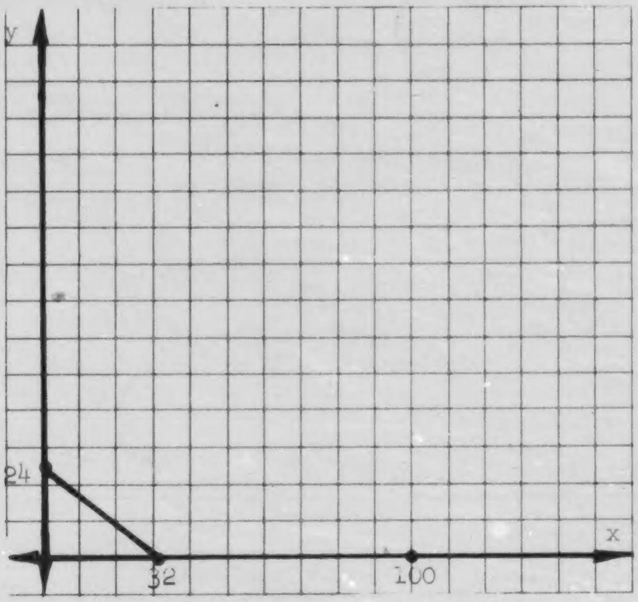
(Show arithmetic solution below.)



_______ % of 75 is 15 .

5. 32% of what number is 24 ?

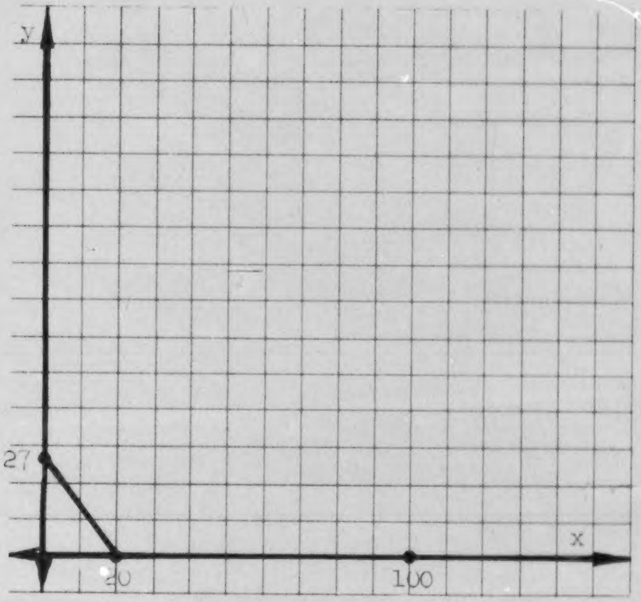
(Show arithmetic solution below.)



32% of _______ is 24 .

6. 20% of what number is 27 ?

(Show arithmetic solution below.)



20% of ________ is 27 .

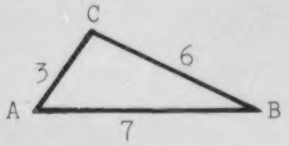
Pre-Test Exercises

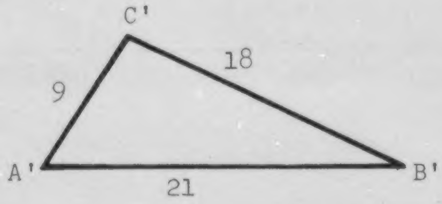
These exercises are like the problems you will have on the chapter test. If you don't know how to do them, read the section again. If you still don't understand, ask your teacher.

1. (Section 12-1.)

Find the scale factor for stretching and the scale factor for shrinking each triangle onto the other.

(a)

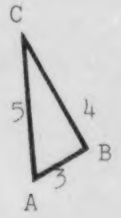


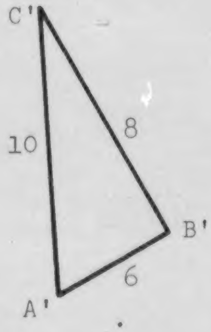


The scale factor for stretching is ________.

The scale factor for shrinking is ________.

(b)



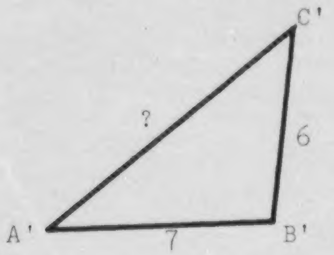


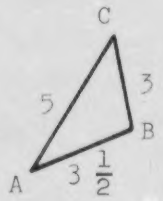
The scale factor for stretching is ________.

The scale factor for shrinking is ________.

2. (Section 12-1.)

Find the scale factor and then find the length of the side of the triangle not given to you.

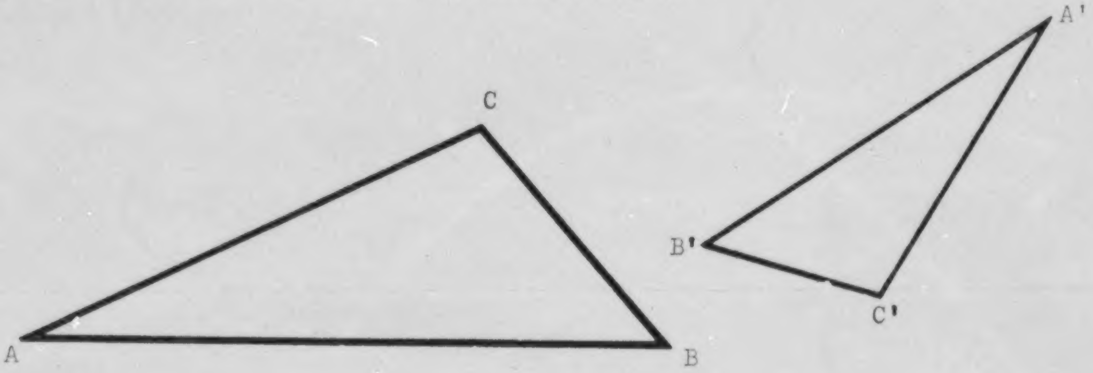




(a) Scale factor is ________.   (b) m $\overline{A'C'}$ = ________.

3. (Section 12-2.)

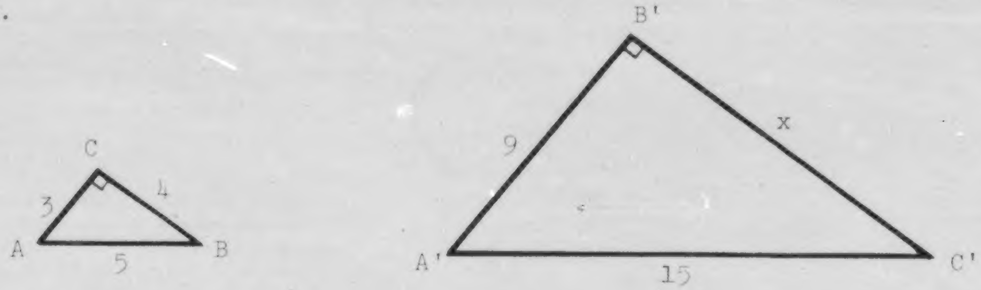
The triangles below are similar. Mark pairs of corresponding sides and pairs of corresponding angles.



4. (Section 12-3.)

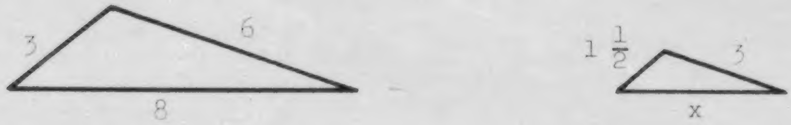
Use equal ratios to find the missing length. The triangles are similar.

(a)


B'
C
9
x
3
4
A
B
A'
C'
5
15

x = ________

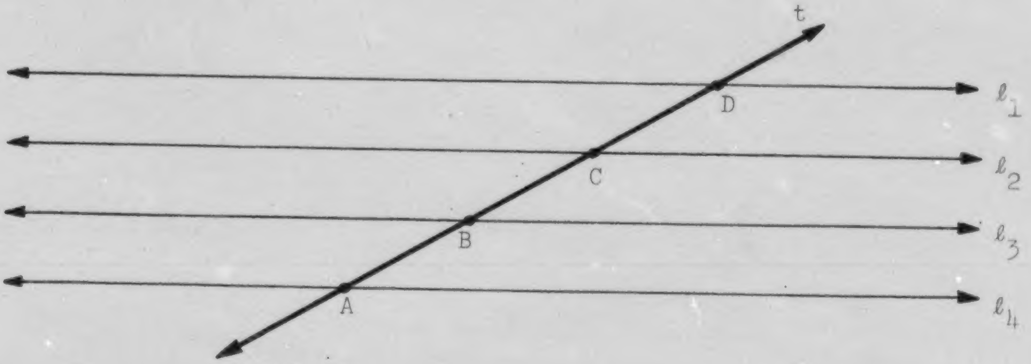
(b)


3
6
$1\frac{1}{2}$
3
8
x

x = ________

5. (Section 12-5.)

In the drawing below, ℓ_1 , ℓ_2 , ℓ_3 , and ℓ_4 are parallel and equally spaced. They are cut by a transversal, t .

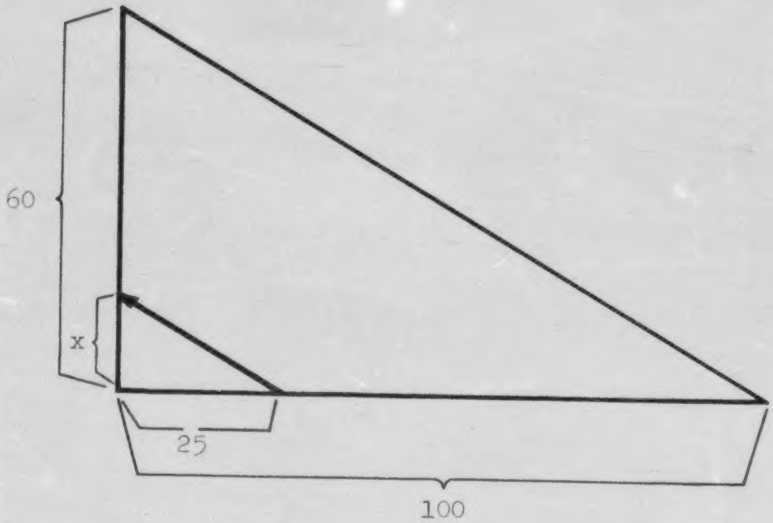

t
D
C
B
A
ℓ_1
ℓ_2
ℓ_3
ℓ_4

What do you know about the measures of segments $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$?

____

6. (Section 12-6.)

Use equal ratios to find the value of x .



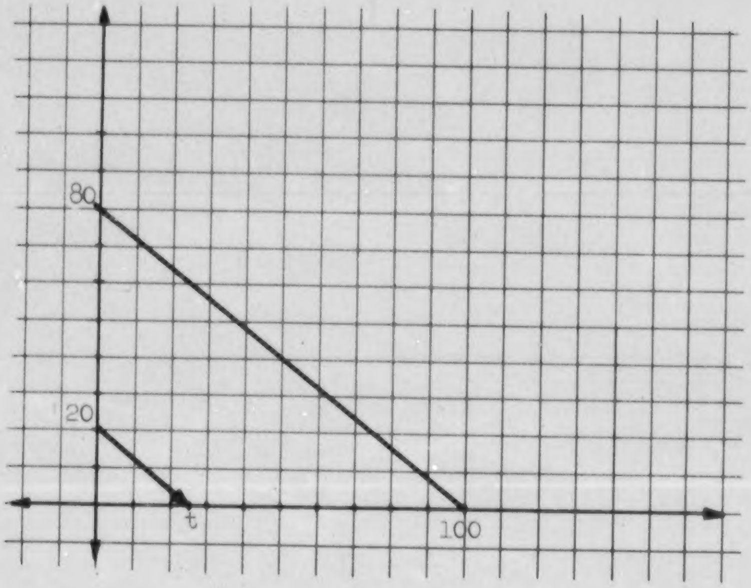
x = __________

7. (Section 12-7.)

$\frac{25}{100} = 25 \cdot \frac{1}{100} =$ ________ %

8. (Section 12-7.)

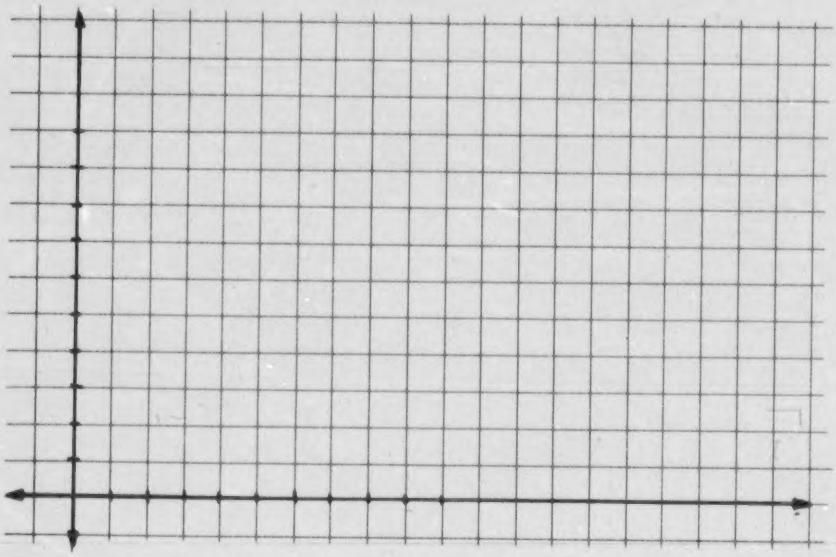
Estimate the value of t .



t is about ________.

9. (Section 12-8.)

On the grid below use similar triangles to estimate the answer to "20% of 90 is what number?".



20% of 90 is ________.

10. (Section 12-8.)

Solve for x .

(a) $\frac{x}{120} = \frac{25}{100}$

x = ____________

(b) $\frac{10}{100} = \frac{9}{x}$

x = ____________

(c) $\frac{x}{100} = \frac{25}{50}$

x = ____________

11. (Section 12-8.)

(a) 3 is what % of 12 ?

3 is ________% of 12 .

(b) 15% of 80 is what number?

15% of 80 is ________.

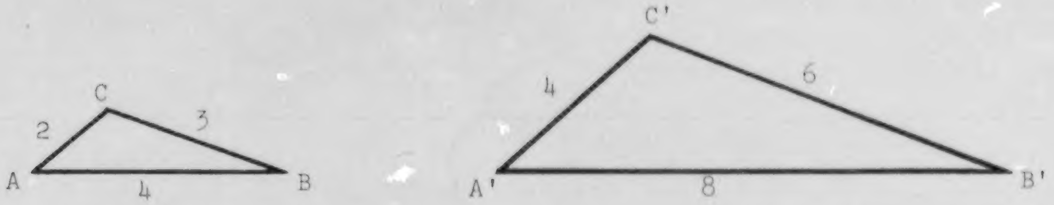
(c) 30% of what number is 36 ?

30% of ________ is 36 .

Test

1. Find the scale factor for stretching and the scale factor for shrinking each triangle onto the other.

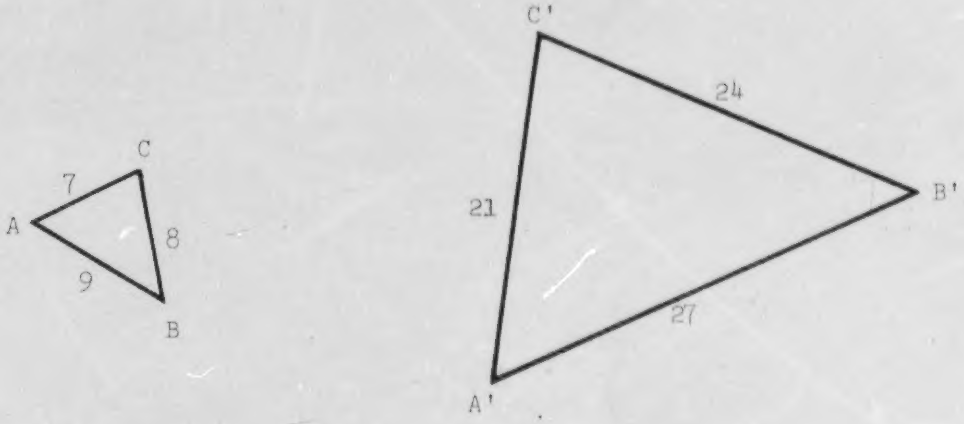
(a)


C
2
3
A
4
B
C'
4
6
A'
8
B'

The scale factor for stretching is ___________.

The scale factor for shrinking is ___________.

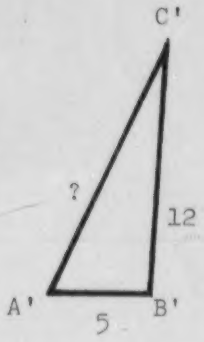
(b)

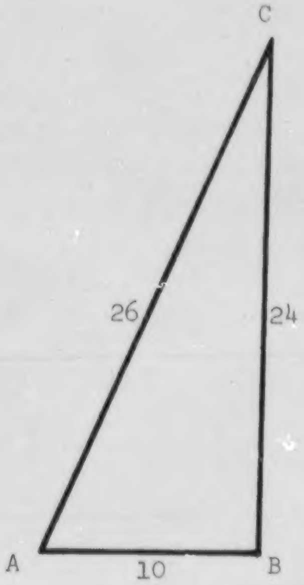

C
7
A
8
9
B
C'
24
21
B'
27
A'

The scale factor for stretching is ___________.

The scale factor for shrinking is ___________.

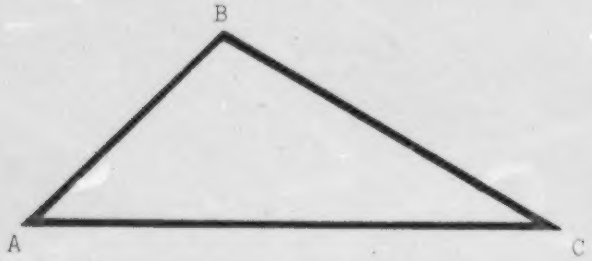
2. Find the scale factor and then find the length of the side not given.

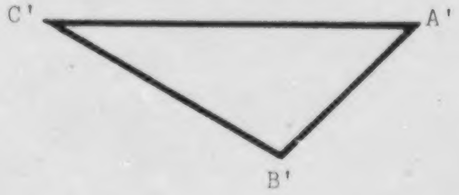




(a) Scale factor is ________. (b) m $\overline{A'C'}$ = ________.

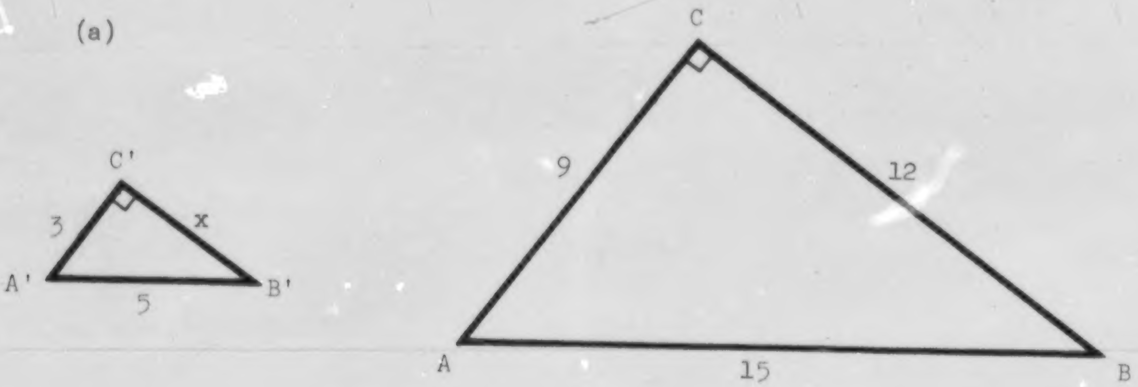
3. The triangles below are similar. Mark pairs of corresponding sides and pairs of corresponding angles.





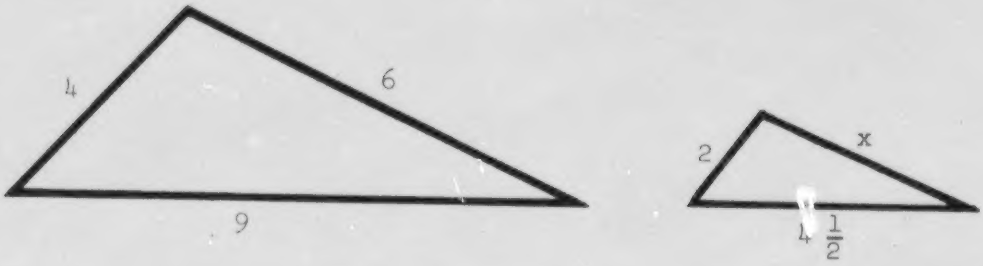
4. Use equal ratios to find missing lengths. The triangles are similar.

(a)


C'
3
x
A'
5
B'
C
9
12
A
15
B

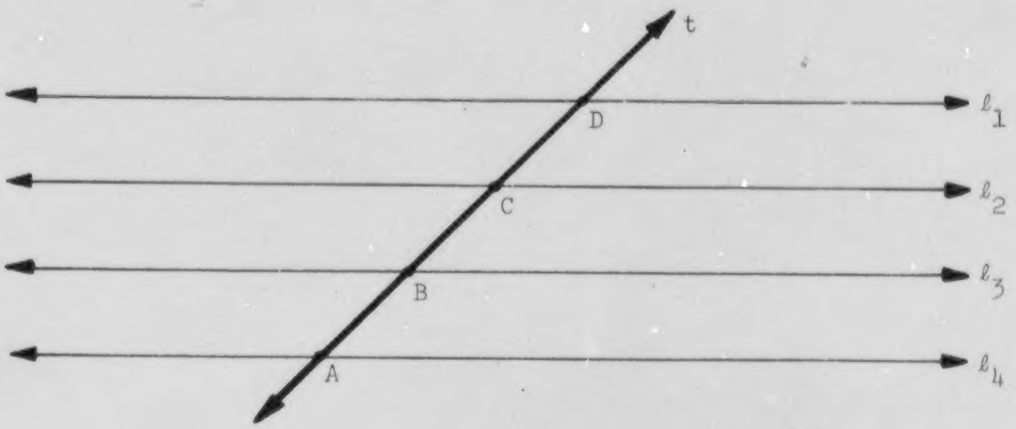
x = ________

(b)


4
6
9
2
x
$4\frac{1}{2}$

x = ________

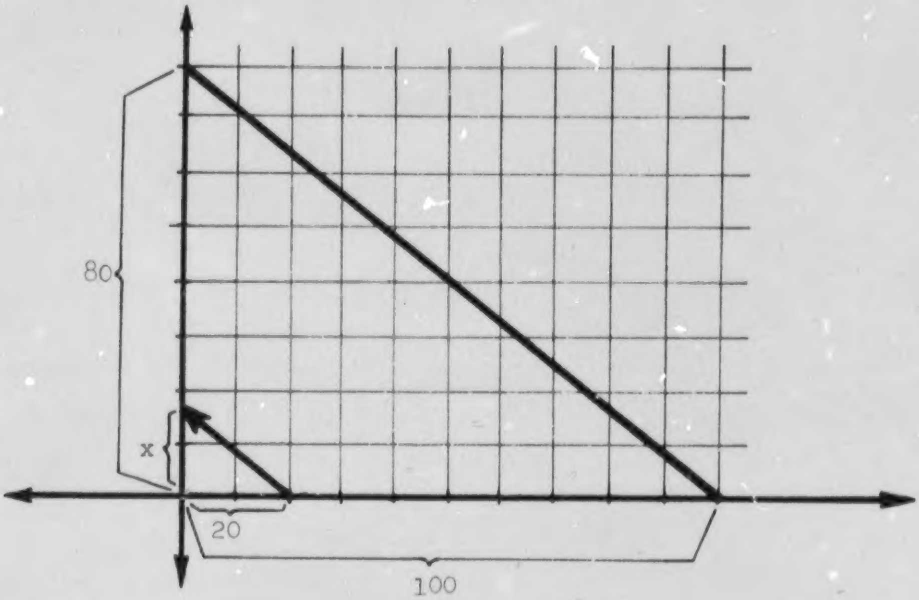
5. In the drawing below, ℓ_1, ℓ_2, ℓ_3, and ℓ_4, are parallel and equally spaced. They are cut by a transversal, t .


t
D
C
B
A
ℓ_1
ℓ_2
ℓ_3
ℓ_4

What do you know about the measures of segments $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$?

____

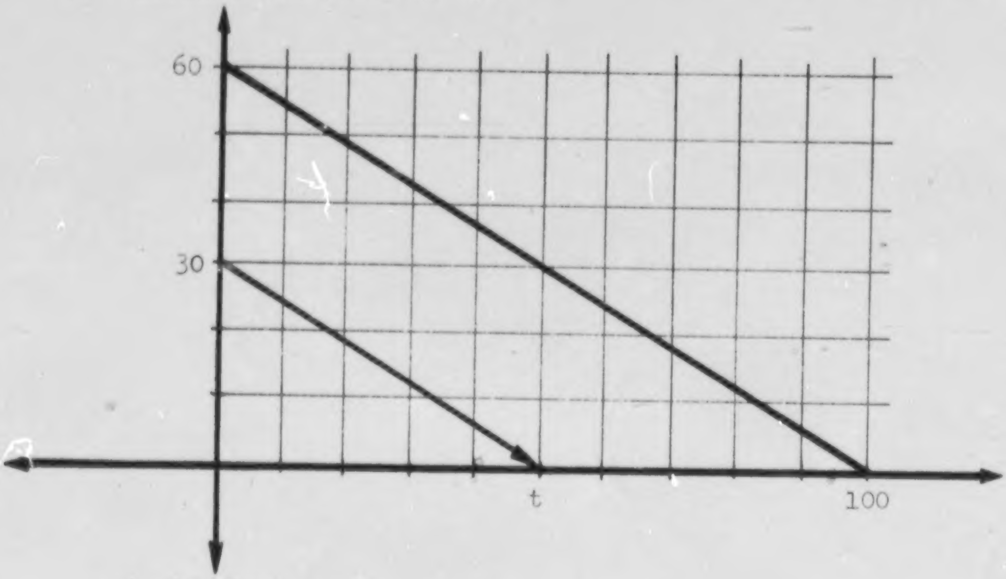
6. Use equal ratios to find the value of x .



x = __________

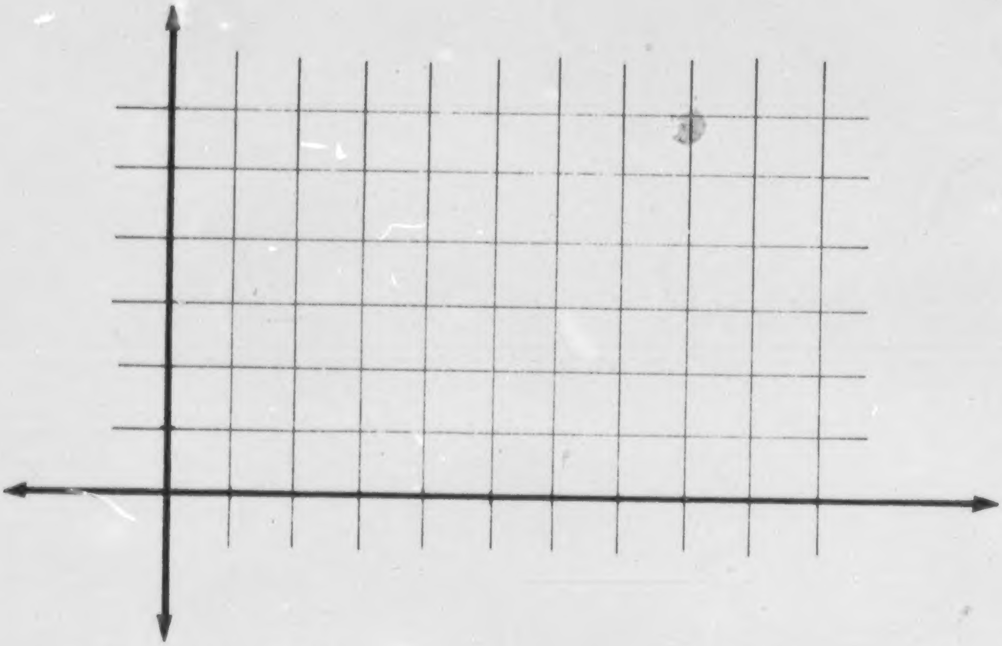
7. $\frac{50}{100} = 50 \cdot \frac{1}{100} =$ __________%

8. Estimate the value of t .



t is about __________.

9. On the grid below, use similar triangles to estimate the answer to "30% of 40 is what number?".



30% of 40 is about ________.

10. Solve for x .

(a) $\frac{x}{160} = \frac{25}{100}$

x = ____________

(b) $\frac{10}{100} = \frac{18}{x}$

x = ____________

(c) $\frac{x}{100} = \frac{15}{60}$

x = ____________

11. (a) 5 is what % of 20 ?

5 is ________ % of 20 .

(b) 25% of 80 is what number?

25% of 80 is ________ .

(c) 40% of what number is 36 ?

40% of ________ is 36 .

Check Your Memory: Self-Test

1. (Section 8-4.)

Fill the blanks.

(a) To undo multiplying by 4 , you multiply by _____.

(b) To undo adding 6 , you __________ __________.

(c) To undo adding $\frac{{}^{-}3}{5}$, you __________ __________.

(d) To undo multiplying by $\frac{3}{8}$, you __________ __________.

2. (Section 8-6.)

Solve these equations.

(a) $7x = {}^{-}35$

$x =$ __________

(b) $x - 11 = {}^{-}5$

$x =$ __________

(c) $\frac{3}{4}x = 6$

$x =$ __________

(d) $\frac{x}{8} = 7$

$x =$ __________

(e) $\frac{2}{3}x - 2 = 4$

$x =$ __________

3. (Section 10-5.)

Put a decimal point in the following numbers so that:

(a) the 5 is in the tenths place	3 4 6 1 8 5
(b) the 8 is in the ones place	7 0 8 9 3
(c) the 4 is in the hundredths place	2 4 5 6 7 1
(d) the 2 is in the thousandths place	5 6 2 4
(e) the 3 is in the tens place	3 0 8

4. (Section 10-11.)

Multiply.

(a) $2.8 \times 100 =$ __________

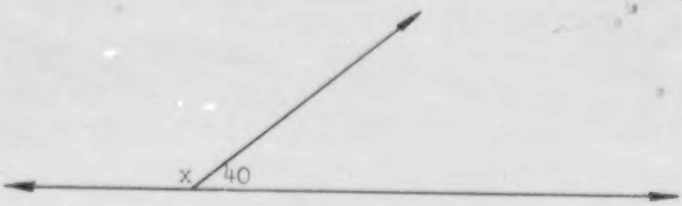
(b) $.372 \times 1000 =$ __________

(c) $.48 \times .25 =$ __________

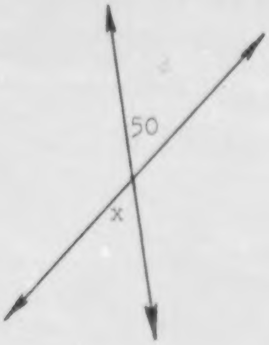
(d) $2 \times .125 =$ __________

5. (Sections 11-5, 11-6, 11-7.)

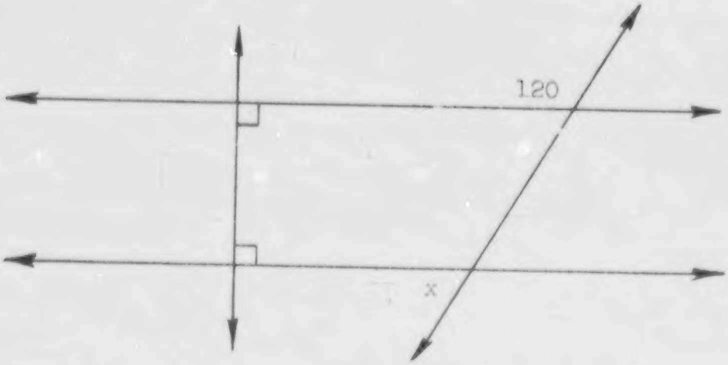
In each of the following figures, find the measure of $\angle x$.



(a) $m \angle x =$ __________



(b) $m \angle x =$ __________



(c) $m \angle x =$ __________

6. (Section 11-4.)

Sketch a figure to show what this looks like.

$\overline{EF} \parallel \overline{HG}$

$\overline{EH} \perp \overline{EF}$

$\overline{HG} \cong \overline{EF}$

$\overline{FG} \perp \overline{HG}$

What kind of figure is this? ______________

Now check your answers on the next page. If you do not have them all right, go back and read the section again.

Answers to Check Your Memory: Self-Test

1. (a) $\frac{1}{4}$
   (b) add $^{-}6$
   (c) add $\frac{3}{5}$
   (d) multiply by $\frac{8}{3}$

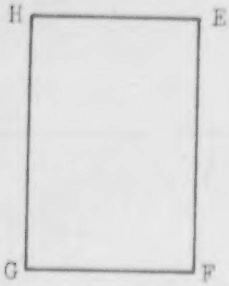
2. (a) $^{-}5$
   (b) 6
   (c) 8
   (d) 56
   (e) 9

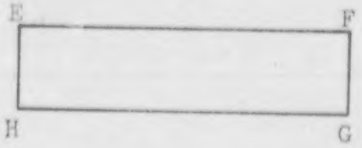
3. (a) 34618.5
   (b) 708.93
   (c) .245671
   (d) .5624
   (e) 30.8

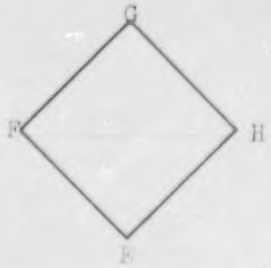
4. (a) 280
   (b) 372
   (c) .1200 = .12
   (d) .250 = .25

5. (a) $m \angle x = 140$
   (b) $m \angle x = 50$
   (c) $m \angle x = 60$

6. The figure is a rectangle. (It is also one kind of parallelogram, and you may have drawn a square.) It may be in many positions, like these:







etc.

However, the angle at H should be opposite the angle at F no matter which way you drew it.

Chapter 13

MORE ABOUT RATIONAL NUMBERS

Chapter 13

MORE ABOUT RATIONAL NUMBERS

Introduction

In your very first experience with numbers, you used numbers to count *things*. You could always find an answer to problems about *things* (and therefore about the numbers you used) simply by moving the things themselves and counting again.

Later you learned to add, subtract, multiply, and divide the *numbers* themselves, and you found that they obeyed certain rules. When you worked with whole numbers, for instance, you knew you could choose two numbers, like 5 and 3 , and add them, and that you would get just one answer, no matter whether you added $5 + 3$ or $3 + 5$.

When you chose two numbers and tried to subtract, however, you found that $5 - 3 = 2$, but for $3 - 5 = ?$, you did not get any answer at all.

In Chapter 5 you learned that the problem $3 - 5 = ?$ does have an answer, $^{-}2$, in the numbers called *integers*. You learned where these numbers are located on the number line.

In the same way, you knew that when you multiplied two whole numbers, the answer was always a whole number, and it didn't matter in which order you multiplied them. $6 \times 3 = 18$ and $3 \times 6 = 18$.

When you tried to divide, however, you found that 6 divided by 3 is a whole number, but 3 divided by 6 is not.

In Chapter 6 you learned about rational numbers and you found that 3 divided by 6 is a *rational number* which may be called $\frac{3}{6}$ or $\frac{1}{2}$ or $\frac{4}{8}$ or many other names, and you learned where the rational numbers are located on the number line.

For a long time you have known how to "compute" using whole numbers. You know how to go about getting the answer when you add and multiply whole numbers, and how to subtract and divide them when an answer is possible.

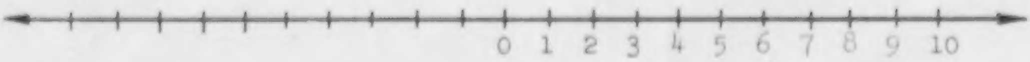
You also know how to add, subtract, and multiply integers, but you have not yet studied division with negative integers. With rational numbers you can multiply and divide, and you know that you can add and subtract all rational numbers, but there are some kinds of problems you still have to learn how to do.

In this chapter, then, you will "fill the gaps" in your ability to compute with integers and rational numbers. You will learn to divide with negative integers and to add or subtract any rational numbers, using fraction names as well as decimal names.

Another Look At Integers

Before you go on to division with negative integers, let's be sure you remember what you learned earlier about these numbers.

Class Discussion

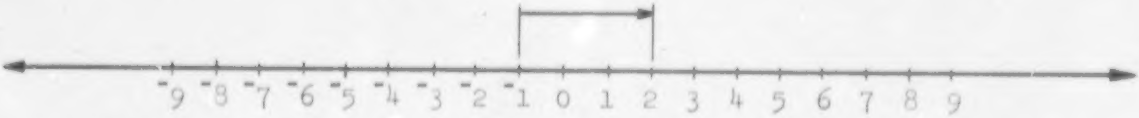


1. On the number line above, the only points that are labeled correspond to whole numbers. You have seen that the numbers to the right of zero are also called ______________ integers. To the left of zero are the ______________ integers, which are the opposites of the positive ones. You have called these "opp 1", "opp 2", and so on. You write ¯1 , ¯2 , ... On the number line above, find the point that is one unit to the left of 0 and label it ¯1 . Label the point that is two units to the left of 0 as ¯2 , etc.

2. The point that is 5 units to the right of 0 corresponds to _________. The point that is 6 units to the left of 0 corresponds to _________.

3. The number _________ is neither positive nor negative.

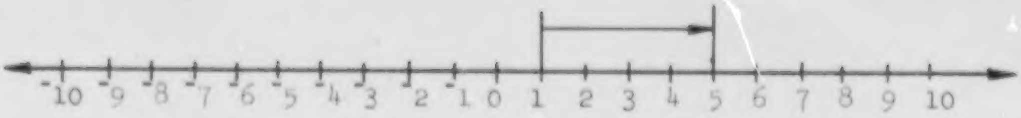
4. Every integer may be represented by an arrow above the number line. The direction the arrow points shows whether the integer is positive or negative. The arrow below shows the integer 3 .



How many units long is the arrow? ___________

Which direction does the arrow point? ___________

5.

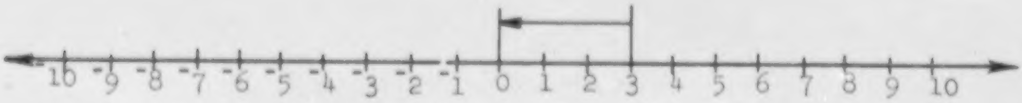

$^-10$ $^-9$ $^-8$ $^-7$ $^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3 4 5 6 7 8 9 10

The arrow above is __________ units long.

It points to the __________.

It represents the integer __________.

6.

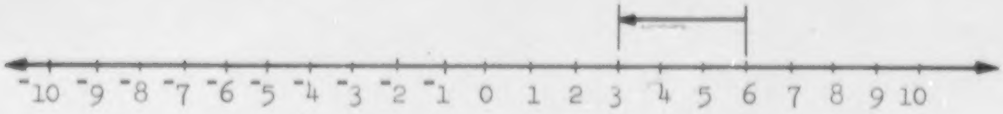

$^-10$ $^-9$ $^-8$ $^-7$ $^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3 4 5 6 7 8 9 10

The arrow above is __________ units long.

It points to the __________.

It represents the integer __________.

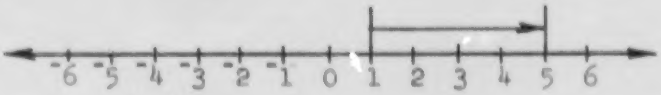
7. To solve $6 + {}^-3$ using an arrow, we start at 6 and draw an arrow 3 units long pointing to the left.


$^-10$ $^-9$ $^-8$ $^-7$ $^-6$ $^-5$ $^-4$ $^-3$ $^-2$ $^-1$ 0 1 2 3 4 5 6 7 8 9 10

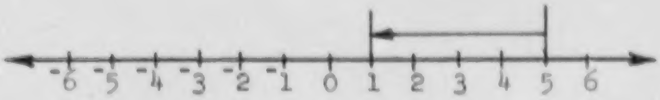
So $6 + {}^-3 =$ __________.

8. Each of these number lines can be used to solve an addition problem.

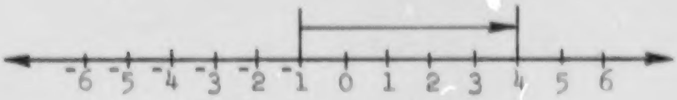
(a)


$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1 2 3 4 5 6

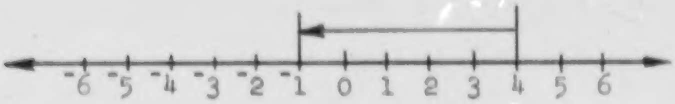
(b)


$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1 2 3 4 5 6

(c)


$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1 2 3 4 5 6

(d)


$^{-}6$ $^{-}5$ $^{-}4$ $^{-}3$ $^{-}2$ $^{-}1$ 0 1 2 3 4 5 6

Write the letter of the number line that goes with each problem, and then give the answer.

(___) $1 + 4 =$ ________ (___) $4 + {}^{-}5 =$ ________

(___) ${}^{-}1 + 5 =$ ________ (___) $5 + {}^{-}4 =$ ________

MULTIPLE CHOICE.

_____ 9. The sum of two positive integers is always

- (a) positive
- (b) negative
- (c) zero
- (d) none of the above

_____ 10. The sum of two negative integers is always

- (a) positive
- (b) negative
- (c) zero
- (d) none of the above

_____ 11. The sum of an integer and its opposite is always

- (a) positive
- (b) negative
- (c) zero
- (d) none of the above

_____ 12. The sum of a positive integer and a negative integer is

- (a) positive if the integer farthest from zero is positive
- (b) negative if the integer farthest from zero is positive
- (c) positive
- (d) negative
- (e) zero

13. (a) In working with whole numbers, you learned that $9 - 7 =$ ________.

(b) In the set of integers, $9 + {}^{-}7 =$ ________.

(c) Subtraction problems may be rewritten as addition problems so that you ______________ the ______________ of the subtrahend.

14. Rewrite each of the following subtraction problems as an addition problem and find the answer.

(a) $4 - 3 =$ ____________ (subtraction problem)

____________________ (addition problem)

(b) $^{-}3 - 2 =$ ____________

____

(c) $4 - {}^{-}3 =$ ____________

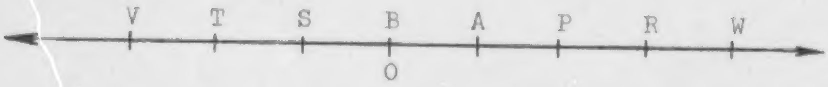
____

(d) $^{-}3 - {}^{-}2 =$ ____________

____

Exercises

1.



On the number line above, letters are used to stand for positive and negative integers. Zero is shown on the line. In the blank beside each letter below, write the letter that stands for its opposite.

(a) A __________

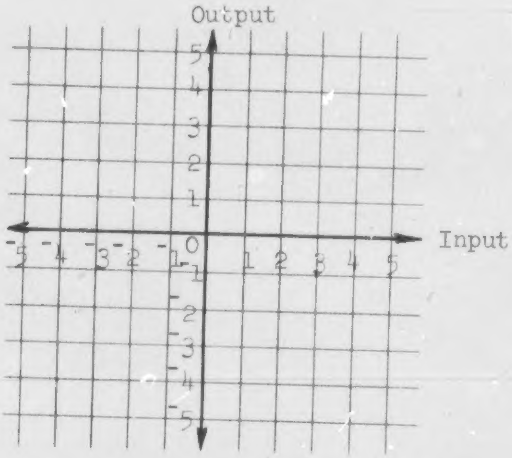
(b) V __________

(c) P __________

(d) B __________

2. The function $f: x \rightarrow \text{opp } x$ is often called the opposite function. Complete the table of inputs and outputs, and graph this function on the coordinate plane below.

$f: x \rightarrow \text{opp } x$	
Input	Output
x	opp x
4	
$^{-}3$	
0	
$^{-}1$	


Output
Input
5
4
3
2
1
0
⁻1
⁻2
⁻3
⁻4
⁻5
⁻5
⁻4
⁻3
⁻2
⁻1
1
2
3
4
5

3. Fill the blanks with $<$ (is less than) or $>$ (is greater than) to make each statement true.

(a) 3 ____ 5
(b) 7 ____ 2
(c) $^{-}6$ ____ 4
(d) 5 ____ $^{-}8$
(e) $^{-}4$ ____ $^{-}3$
(f) $^{-}16$ ____ $^{-}1$

4. Find the answer to each problem below. If it is an addition problem, you will not need the blank below the problem. If it is a subtraction problem, rewrite the problem to show that you add the opposite of the subtrahend.

(a) $3 + 5 =$ ____
____

(b) $3 + {}^{-}5 =$ ____
____

(c) $4 - 3 =$ ____
____

(d) $4 - {}^{-}3 =$ ____
____

(e) $6 - {}^{-}12 =$ ____
____

(f) $8 + {}^{-}7 =$ ____
____

(g) $^{-}4 + 7 =$ ____
____

(h) $^{-}8 - {}^{-}5 =$ ____
____

(i) $^{-}4 - 7 =$ ____
____

(j) $^{-}3 - {}^{-}6 =$ ____
____

(k) $14 + {}^{-}14 =$ ____
____

(l) $7 - 7 =$ ____
____

(m) $^{-}9 + 9 =$ ____
____

(n) $^{-}9 - {}^{-}9 =$ ____
____

Multiplying and Dividing with All the Integers

Because you can think of multiplication as repeated addition, you know that:

$$\begin{aligned} {}^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 &= 4 \cdot {}^{-}3 \\ {}^{-}3 + {}^{-}3 + {}^{-}3 + {}^{-}3 &= {}^{-}12 \\ \text{so} \qquad 4 \cdot {}^{-}3 &= {}^{-}12 \,. \end{aligned}$$

With integers, just as with whole numbers, it doesn't make any difference in which order you multiply two numbers. So you see that since $4 \cdot {}^{-}3 = {}^{-}3 \cdot 4$, and $4 \cdot {}^{-}3 = {}^{-}12$, it must be true that ${}^{-}3 \cdot 4$ also is ${}^{-}12$.

Class Discussion

1. When you multiply two positive integers, the answer is ____________.
2. When you multiply a negative integer by a positive integer, the answer is _______________.
3. When you multiply a positive integer by a negative integer, the answer is _______________.
4. When you multiply two negative integers, the answer is ____________.

In Chapter 5 you learned that the answer, in Question 4, is "positive". You found this out by graphing the "doubling" function and by making a multiplication table. You may have found it hard to believe, however.

Now let's make this clearer and at the same time learn how to divide with negative integers.

In all your work with numbers, you have found that a number may have many different names. The number 12, for instance, may be named using addition $(9 + 3)$, subtraction $(13 - 1)$, multiplication (6×2), or division $(\frac{24}{2})$. However, each one of these names "belongs to" the number 12 only. If you want to name some number that is not 12, you can't call it $9 + 3$, because $9 + 3$ is always 12. (It would be very awkward if this were not so. Think about it.)

You also know that multiplication statements about numbers can be rewritten as division statements using exactly the same numbers.

$3 \cdot 4 = 12$ so $\frac{12}{3}$ must be 4

$3 \cdot {}^{-}4 = {}^{-}12$ so $\frac{{}^{-}12}{3}$ must be ________

${}^{-}3 \cdot 4 = {}^{-}12$ so $\frac{{}^{-}12}{{}^{-}3}$ must be ________

The answer to this question

$${}^{-}3 \cdot {}^{-}4 = ?$$

can't be ${}^{-}12$, because then we would have to say that

$$\frac{{}^{-}12}{{}^{-}3} = {}^{-}4 ,$$

and in that case $\frac{{}^{-}12}{{}^{-}3}$ would name two different numbers, 4 and ${}^{-}4$.

We know that this can't be true, so

$${}^{-}3 \cdot {}^{-}4 = 12 ,$$

and $\frac{12}{{}^{-}3} =$ ________ .

From the last problem, you are ready to fill the blanks in these statements.

5. When the integers you multiply are both positive or both negative, the answer is ____________.

6. When you divide integers that are both positive or both negative, the answer is ____________.

7. When you divide integers, if one is positive and the other is negative, the answer is ____________.

Exercises

1. $7 \cdot 15 =$ ________

2. $\frac{105}{7} =$ ________

3. ${}^-8 \cdot 2 =$ ________

4. $\frac{{}^-16}{2} =$ ________

5. $9 \cdot {}^-3 =$ ________

6. $\frac{{}^-27}{{}^-3} =$ ________

7. ${}^-5 \cdot {}^-6 =$ ________

8. $\frac{30}{{}^-5} =$ ________

9. ${}^-5 \cdot 9 =$ ________

10. $\frac{{}^-72}{12} =$ ________

11. ${}^-14 \cdot {}^-4 =$ ________

12. $\frac{{}^-81}{{}^-27} =$ ________

13. $\frac{{}^-64}{16} =$ ________

14. $5 \cdot {}^-19 =$ ________

15. $\frac{48}{{}^-12} =$ ________

16. $4 \cdot 28 =$ ________

17. ${}^-15 \cdot {}^-15 =$ ________

18. $\frac{{}^-625}{25} =$ ________

19. $\frac{900}{30} =$ ________

20. $\frac{169}{{}^-13} =$ ________

Do your work in the space at the right.

21. $\frac{352}{{}^-2} =$ ________

22. ${}^-25 \cdot 46 =$ ________

23. $\frac{{}^-513}{{}^-3} =$ ________

24. $\frac{{}^-729}{9} =$ ________

25. ${}^-54 \cdot {}^-13 =$ ________

Another Look at Rational Numbers

Now that you can divide both positive and negative integers when the answer is an integer, you can apply what you know to other rational numbers.

Class Discussion

1. $\frac{16}{^-4}$ = ________ and $\frac{^-16}{4}$ = ________ .

The integer named by both $\frac{16}{^-4}$ and $\frac{^-16}{4}$ is the same. So you can say that

$$\frac{16}{^-4} = \frac{^-16}{4} .$$

Parentheses ( ) mean, "Do this first". If you have a problem like $3 + (\frac{16}{4}) = ?$, you think, " $\frac{16}{4} = 4$, so I add $3 + \underline{4} = 7$ ".

We use parentheses in the same way with the raised dash on the outside: $^-(\ )$. In this case, you think, "Take the opposite of the number that is inside the parentheses."

$\frac{16}{4}$ = ________ so $^-(\frac{16}{4})$ = ________

You see that $\frac{^-16}{4}$ and $\frac{16}{^-4}$ and $^-(\frac{16}{4})$ are all names for ________.

Therefore, you know that

$$\frac{^-16}{4} = \frac{16}{^-4} = {}^-(\frac{16}{4}) .$$

2. Rewrite each of the following as a fraction in two different ways. The first one is done for you.

(a) $\frac{^-1}{8}$ $\underline{\frac{1}{^-8}}$ $\underline{^-(\frac{1}{8})}$

(b) $\frac{^-9}{2}$ ________ ________

(c) $\frac{6}{^-3}$ ________ ________

(d) $^-(\frac{5}{6})$ ________ ________

(e) $\frac{8}{^-9}$ ________ ________

(f) $\frac{^-4}{^-5}$ ________ ________

(g) $^-(\frac{2}{3})$ ________ ________

Although we know that a negative rational number can be named in the ways we have discussed, it is often important in solving problems to use only one certain way. For instance, you have added rational numbers like this:

$$\begin{aligned}\frac{6}{3} + \frac{6}{3} &= \frac{6 + 6}{3} \\ &= \frac{12}{3} \\ &= 4 .\end{aligned}$$

Suppose, now, you must add $\frac{6}{3} + \frac{6}{{}^{-}3}$. Of course, in this problem you could use the integer name for each number and add: $2 + {}^{-}2 = 0$.

Let's see how you can do it using the fraction names.

$$\frac{6}{3} + \frac{6}{{}^{-}3} = \ ?$$

What denominator would you use? Instead of writing $\frac{6}{-3}$, write $\frac{{}^{-}6}{3}$, and then you can find the answer in the usual way.

$$\begin{aligned}\frac{6}{3} + \frac{{}^{-}6}{3} &= \frac{6 + {}^{-}6}{3} \\ &= \frac{0}{3} \\ &= 0\end{aligned}$$

3. Show how to rewrite each of the following problems so that tne denominator is the same. Then find the answer in simplest form.

(a) $\frac{1}{4} + \frac{1}{{}^{-}4} = \frac{\quad + \quad}{4}$

$=$

(b) $\frac{4}{{}^{-}5} + \frac{2}{5} =$

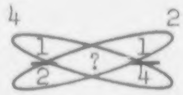
$=$

(c) $\frac{{}^{-}9}{10} + \frac{7}{{}^{-}10} =$

$=$

4. Another kind of problem comes up when you try to compare two rational numbers. Let's recall what you did with positive numbers in Chapter 6.

Example. $\frac{1}{2}$ and $\frac{1}{4}$

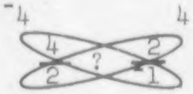


4 is greater than 2, so you know that $\frac{1}{2}$ is greater than $\frac{1}{4}$.

Let's try this method to compare $\frac{4}{2}$, which is another name for _______, and $\frac{2}{^{-}1}$, which is another name for _________.

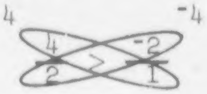
You know that 2 is greater than $^{-}2$, so $\frac{4}{2} > \frac{2}{^{-}1}$.

However, if you write



it looks as if it were the other way around, because $^{-}4$ is less than 4.

You get the correct result when you rename $\frac{2}{^{-}1}$ as $\frac{^{-}2}{1}$.



To avoid mistakes when you work with fraction names for negative numbers, write the negative symbol in the numerator rather than in the denominator. The form used should be $\frac{^{-}a}{b}$.

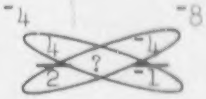
5. (a) $\frac{4}{2} =$ _________

(b) $\frac{^{-}4}{^{-}2} =$ _________

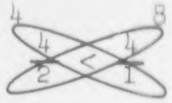
(c) Therefore, $\frac{4}{2} = \frac{^{-}4}{^{-}2}$. $\frac{^{-}4}{^{-}2}$ is the name of a positive rational number. Certainly you would use $\frac{4}{2}$ instead of $\frac{^{-}4}{^{-}2}$ if you were going to add: $\frac{4}{2} + \frac{^{-}4}{^{-}2} = ?$

$$\begin{aligned} \frac{4}{2} + \frac{4}{2} &= \frac{4+4}{2} \\ &= \frac{8}{2} \\ &= 4. \end{aligned}$$

6. If you want to compare $\frac{4}{2}$ and $\frac{^-4}{^-1}$, you could write:



$^-4$ is greater than $^-8$, so it looks as if $\frac{4}{2}$ is greater than $\frac{^-4}{^-1}$. But this is impossible, because $\frac{4}{2}$ = _____ and $\frac{^-4}{^-1}$ = _____ and 2 is not greater than 4 . If you use the name $\frac{4}{1}$ instead of $\frac{^-4}{^-1}$, you get the right answer:



To be sure you get the right answer in such problems, write all rational numbers which are positive without any negative symbols. Write $\frac{a}{b}$, not $\frac{^-a}{^-b}$.

Exercises

1. Compare these rational numbers. Rewrite the fractions if necessary. Then put < or > in the blank.

(a) $\frac{^-15}{13}$ ________ $\frac{3}{^-4}$

(b) $\frac{^-7}{^-8}$ ________ $\frac{6}{11}$

(c) $\frac{18}{^-5}$ ________ $\frac{4}{^-3}$

(d) $\frac{^-9}{8}$ ________ $\frac{2}{^-3}$

(e) $\frac{^-1}{^-3}$ ________ $\frac{1}{4}$

2. Use what you know about rewriting rational numbers to find the sums of the two numbers in each problem below. Give your answer in simplest form.

Example. $\frac{^-2}{5} + \frac{4}{^-5} = ?$

$$\begin{aligned} \frac{^-2}{5} + \frac{4}{^-5} &= \frac{^-2}{5} + \frac{^-4}{5} \\ &= \frac{^-2 + {^-4}}{5} \\ &= \frac{^-6}{5} \qquad \text{(answer)} \end{aligned}$$

(a) $\frac{7}{^-8} + \frac{^-1}{8} =$ __________

(b) $\frac{^-3}{5} + \frac{^-2}{^-5} =$ __________

(c) $\frac{^-1}{^-6} + \frac{7}{6} =$ __________

Multiples

Much of what you can do with any kind of number depends on what you know about whole numbers. You depend on the whole numbers to help you work with rational numbers. Learning about multiples of whole numbers, for instance, will help you add rational numbers.

Your multiplication table for whole numbers can also be called a table of products, because it tells you the product of any two numbers from 0 through 30 . We can use still another name for this useful table. It is a table of multiples.

If you multiply two whole numbers, the result is a multiple of each of them.

You get 14 when you multiply 2 by 7 , so 14 is a multiple of 2 and a multiple of 7 .

You also get 14 when you multiply 1 by 14 , so 14 is also a multiple of 1 and a multiple of 14 .

As you see, 14 is a multiple of four different numbers: 1 , 2 , 7 , and 14 .

Class Discussion

Look at the "2" row in your multiplication table. On the first page, you see

0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30

Each of these numbers is a multiple of 2 , because

$0 = 2 \times 0$

$2 = 2 \times 1$

$4 = 2 \times 2$

$6 = 2 \times 3$

$8 = 2 \times 4$

$10 = 2 \times 5$

and so on.

On the second page, you find the multiples of 2 that you get from multiplying 2 by each number from 16 to 30 .

1. The multiples of _____ are all even numbers. If your table had a third page, you could find more multiples of 2 . The next multiple after 60 is 62 .

2. Which multiples of 2 come after 62 ?

   _____ _____ _____ _____ _____ _____ _____ _____ _____

3. What multiple of 2 do you get when you multiply 2 by 400 ? _______ What multiple of 2 do you get when you multiply 2 by 10,000 ? _________ What multiple of 2 do you get when you multiply 2 by 3,000,000,000 ? _________________
   As you know, there is no last or largest whole number, so there is no last or largest multiple of 2 .

4. List the first ten multiples of 3 .

   _____ _____ _____ _____ _____ _____ _____ _____ _____ _____

5. List the first ten multiples of 7 .

   _____ _____ _____ _____ _____ _____ _____ _____ _____ _____

6. List the first ten multiples of 5 .

   _____ _____ _____ _____ _____ _____ _____ _____ _____ _____

7. The number that appears in every row of multiples is _______ . Because any number times zero is zero, it is a multiple of every number, and because it's always in the list, we usually don't bother to write 0 when we list the multiples of a number.

8. After 0 , what is the first multiple of 1 ? _______

9. After 0 , what is the first multiple of 2 ? _____ Of 3 ? _____ Of 4 ? _____ Of 5 ? _____ Of 16 ? _____ Of 29 ? _____

10. The first multiple of any number is the _____________ itself. Every number is a _____________ of itself.

Now look through your table carefully.

11. How many times do you find the number 2 as a multiple?

____

12. The number 2 is a multiple of only two numbers, 2 and ______.

13. How many times do you find the number 3 as a multiple? ______________
3 is a multiple of only two numbers, ______ and ______.

14. In Chapter 4, you learned about prime numbers. 2 and 3 are prime numbers. They are multiples only of themselves and 1 . List five other prime numbers from your table. (Note: They are numbers that appear as multiples exactly twice.)

_____ _____ _____ _____ _____

Exercises

1. Draw a ring around each number that does not appear at all on the first page of your table.

31 33 37 39 41 43 47 49 51 53 57 59
61 63 67 69 71 73 77 79 81 83 87 89

2. You have circled the prime numbers between 30 and 90 . Write each number that you did not circle and show that it is a multiple of two numbers besides itself and 1 .

Example. 33 = 3 × 11

______________________ ______________________
______________________ ______________________
______________________ ______________________
______________________ ______________________
____

3. What number is a multiple of every number? __________

4. Show that 24 is a multiple of 1 , 2 , 3 , 4 , 6 , 8 , 12 , and 24 .

(a) 24 = 1 × ________

(b) 24 = 2 × ________

(c) 24 = 3 × ________

(d) 24 = 4 × ________

(e) 24 = 6 × ________

(f) 24 = 8 × ________

(g) 24 = 12 × ________

(h) 24 = 24 × ________

5. Show that 16 is a multiple of 1 , 2 , 4 , 8 , and 16 .

(a) 16 = 1 × ________

(b) 16 = 2 × ________

(c) 16 = 4 × ________

(d) 16 = 8 × ________

(e) 16 = 16 × ________

6. Show that 12 is a multiple of 1 , 2 , 3 , 4 , 6 , and 12 .

(a) 12 = 1 × ________

(b) 12 = 2 × ________

(c) 12 = 3 × ________

(d) 12 = 4 × ________

(e) 12 = 6 × ________

(f) 12 = 12 × ________

7. Show that 15 is a multiple of 1 , 3 , 5 , and 15.

(a) 15 = 1 × ________

(b) 15 = 3 × ________

(c) 15 = 5 × ________

(d) 15 = 15 × ________

Common Multiples

In your multiplication table, the numbers in the "1" row are just like the numbers at the top of the table. From this you know that 0 is a multiple of 1 , 1 is a multiple of 1 , 2 is a multiple of 1 , 3 is a multiple of 1 , and so on. Every whole number is a multiple of 1 .

Class Discussion

1. (a) A number that is a multiple of _________ is called an even number.

   (b) Even numbers are numbers which have, as their last digit, _____, _____, _____, _____, or _____.

   (c) List the first ten multiples of 2 . _____ _____ _____ _____ _____ _____ _____ _____ _____ _____

2. List the first ten multiples of 3 . _____ _____ _____ _____ _____ _____ _____ _____ _____ _____

3. Which numbers are in both lists?

   _____, _____, and _____

Because these numbers are common to both lists, they are called common multiples of 2 and 3 . What is the smallest common multiple of 2 and 3 ? _________ This is called the least common multiple of 2 and 3 , and is usually shortened to L.C.M.

Every common multiple of 2 and 3 is also a multiple of 6 . Is there a greatest common multiple of 2 and 3 ? _________

4. Is 24 a common multiple of 3 and 4 ? _________ What is the smallest number that is in both the "3" row and the "4" row? _________ The L.C.M. is _________.

Suppose you want to find a common multiple of two numbers like 6 and 8 when you don't have your multiplication tables with you. The adding method is an easy way.

Write down both numbers.

6 8

6 is less than 8 . Add 6 to 6 .

6 ⑧
(6 + 6) = ⑫

Look at the bottom numbers. 8 is less than 12 . Add 8 to 8 .

6 8
(6 + 6) = ⑫ ⑯ = (8 + 8)

Compare the bottom numbers again. The one in the 6 column is smaller than the one in the 8 column. Add 6 to 12 .

6 8
(6 + 6) = 12 ⑯ = (8 + 8)
(6 + 12) = ⑱

Compare. $16 < 18$. Add 8 to 16 .

6 8
(6 + 6) = 12 16 = (8 + 8)
(6 + 12) = ⑱ ㉔ = (8 + 16)

Compare. $18 < 24$. Add 6 to 18 .

6 8
(6 + 6) = 12 16 = (8 + 8)
(6 + 12) = 18 ㉔ = (8 + 16)
(6 + 18) = ㉔

24 = 24 , so 24 is a common multiple -- in fact, the least common multiple of 6 and 8 . All the common multiples of 6 and 8 are multiples of 24 ; that is, 48 , 72 , 96 , 120 , 144 , etc.

5. Use this method to find a common multiple of 9 and 12 .

9 12
(9 + 9) = ___ ___ = (12 + 12)
(9 + ___) = ___ ___ = (12 + ___)
(9 + ___) = ___

The least common multiple of 9 and 12 is _______. All of the common multiples of 9 and 12 are multiples of _______.

Exercises

1. Find the least common multiple for each pair of numbers. (You do not have to write (10 + 10) , etc., at the side to show what you add.)

(a) 10 15
___ ___
___ L.C.M. is _________.

(b) 8 10
___ ___
___ ___
___ ___
___ L.C.M. is _________.

(c) 15 20
___ ___
___ ___
___ L.C.M. is _________.

(d) 9 15
___ ___
___ ___
___
___ L.C.M. is _________.

(e) 12 8

____ ____

____ L.C.M. is _________.

(f) 6 9

____ ____

____ L.C.M. is _________.

(g) 21 14

____ ____

____ L.C.M. is _________.

2. (a) All of the common multiples of 10 and 15 are also multiples of _________.

List the next 3 common multiples of 10 and 15 .

________, ________, ________

(b) All of the common multiples of 8 and 10 are also multiples of _________.

(c) Give the first three common multiples of 9 and 15 .

________, ________, ________

Class Discussion

1. If you use the adding method to find the least common multiple of 2 and 8 , it goes like this:

2 8

$2 < 8$, so you add 2 to 2 .

2 8
4

$4 < 8$, so you add ______________ .

2 8
4
6

$6 < 8$, so you add ______________ .

2 8
4
6
8

You now have $8 = 8$, and you know that the L.C.M. of 2 and 8 is 8 .

It is easier to use a different method to find the least common multiple of 2 and 8 .

You have probably realized that *if, in a pair of numbers, one of them is a multiple of the other, then the larger one is the least common multiple of the pair*.

2. The least common multiple of 2 and 8 is _______, because _______ is a multiple of 2 .

3. The least common multiple of 12 and 6 is _______, because _______ is a multiple of 6 .

4. The least common multiple of 20 and 4 is _______, because 20 is a multiple of _______.

5. The least common multiple of 8 and 24 is _______, because _______ is a multiple of _______.

6. Now let's use the adding method to find the least common multiple of 5 and 7 .

5	7
___	___
___	___
___	___
___	___
___	
___	

The least common multiple of 5 and 7 is ______, but
5 × 7 = ______!

---

7. What is the least common multiple of each of these pairs?

2 and 3 ? ________ (And 2 × 3 = ______)
2 and 5 ? ________ (And 2 × 5 = ______)
2 and 7 ? ________ (And 2 × 7 = ______)
3 and 5 ? ________ (And 3 × 5 = ______)
3 and 7 ? ________ (And 3 × 7 = ______)

What special kind of numbers are 2 , 3 , 5 , and 7 ? __________

When you want to find the least common multiple of two prime numbers, the easiest way is just to multiply the two numbers.

8. In the pair of numbers 9 and 4 , neither number is prime. Find the least common multiple of 9 and 4 .

9	4
___	___
___	___
___	___
	___
	___
	___
	___
	___

The least common multiple of 9 and 4 is ______, and
9 × 4 = ______.

To see why this happens, think what prime numbers are multiplied together to get 9 and to get 4 . In Chapter 4 you learned that the prime factorization of 9 is 3×3 and the prime factorization of 4 is 2×2 . The numbers 9 and 4 do not have any factors that are alike, so we say that they are relatively prime. That is, they are not prime themselves, but they are prime in relation to each other. Relatively prime numbers do not have any common factors except 1 .

On the other hand, 9 and 12 are not relatively prime because 3 is a factor of both. $9 = 3 \times 3$ and $12 = 2 \times 2 \times 3$, so the least common multiple of 9 and 12 is not 9×12 .

The L.C.M. of 9 and 12 is $2 \times 2 \times 3 \times 3 = 36$.

Exercises

For the following pairs of numbers, use the method that is easiest for you to find the least common multiple.

1. 5 and 10 L.C.M. __________

2. 4 and 12 L.C.M. __________

3. 3 and 4 L.C.M. __________

4. 8 and 9 L.C.M. __________

5. 9 and 24 L.C.M. __________

6. 2 and 9 L.C.M. __________

7. 15 and 18 L.C.M. __________

8. 7 and 9 L.C.M. __________

9. 16 and 12 L.C.M. __________

10. 10 and 24 L.C.M. __________

11. 11 and 3 L.C.M. __________

12. 14 and 21 L.C.M. __________

Adding Rational Numbers

Now you will use common multiples to add any two rational numbers, even if their denominators are not the same.

Class Discussion

When you add $\frac{1}{4} + \frac{1}{4}$, you write $\frac{1}{4} + \frac{1}{4} = \frac{1 + 1}{4}$, and then simplify your answer if possible, like this:

$$\begin{aligned} \frac{1}{4} + \frac{1}{4} &= \frac{1+1}{4} \\ &= \frac{2}{4} \\ &= \frac{1}{2} \end{aligned}$$

1. To make sure you remember how to add rational numbers, find the sums for each of the problems below.

(a) $\frac{2}{8} + \frac{3}{8} = \frac{\quad + \quad}{8}$

$= \frac{\qquad}{8}$

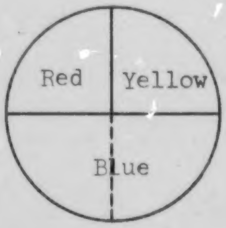
(b) $\frac{2}{3} + \frac{1}{3} = \frac{\quad + \quad}{3}$

$= \frac{\qquad}{3}$ (Simplify your answer: ________.)

(c) $\frac{5}{8} + \frac{^{-}1}{8} = \frac{\quad + \quad}{8}$

$= \frac{\qquad}{8}$ (Simplify your answer: ________.)

2. In Chapter 7, you saw that sometimes you had to use a different name for a rational number in order to add it to another number.



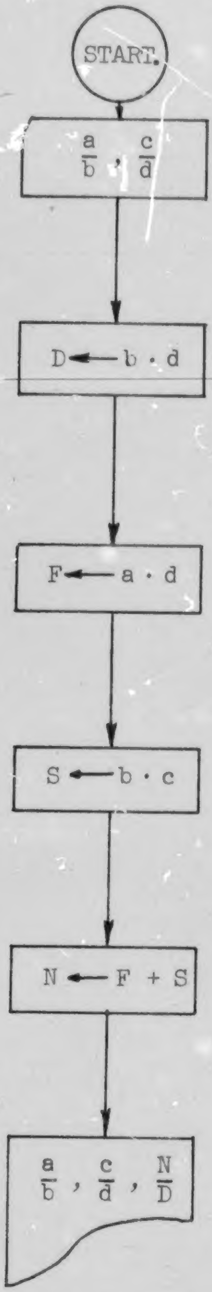
In this spinner, $P(\text{red}) = \frac{1}{4}$ and $P(\text{blue}) = \frac{1}{2}$. To find P(either red or blue) you did not know how to add $\frac{1}{2} + \frac{1}{4}$, so you used a different name for $\frac{1}{2}$. You saw that P(blue) could be written as _______, so you added

$$\frac{2}{4} + \frac{1}{4} = \frac{\quad + \quad}{4}$$

$$=$$

So, P(either red or blue) = _________.

3. Here is a flow chart you may use to add any two rational numbers. Use inputs of $\frac{3}{8}$ and $\frac{3}{4}$ and show the output below.


START.
$\frac{a}{b}$, $\frac{c}{d}$
$D \leftarrow b \cdot d$
$F \leftarrow a \cdot d$
$S \leftarrow b \cdot c$
$N \leftarrow F + S$
$\frac{a}{b}$, $\frac{c}{d}$, $\frac{N}{D}$

Output: $\frac{3}{8}$, $\frac{3}{4}$, __________ You have added $\frac{3}{8} + \frac{3}{4}$ and found that the sum is $\frac{36}{32}$, although $\frac{36}{32}$ is not in simplest form.

You can see how this was done if you rename $\frac{3}{8}$ and $\frac{3}{4}$.

$\frac{3}{8} \cdot \frac{4}{4} = \frac{\quad}{32}$ (write the numerator) and $\frac{3}{4} \cdot \frac{8}{8} = \frac{\quad}{32}$ (write the numerator).

So you add $\frac{12}{32} + \frac{24}{32} = \frac{\quad + \quad}{32}$

$$= \underline{\qquad\qquad}.$$

Now you can see how to use common multiples. When you add two rational numbers, they must have the same denominator. The problem, then, is to find some common denominator and use it to find new names for the numbers.

When you multiply any two numbers together, the answer is a multiple of each of them. In the box $\boxed{D \leftarrow b \cdot d}$ you got a common denominator by finding a common multiple of the two denominators, 8 and 4.

$$8 \cdot 4 = \textcircled{32}$$

In $\boxed{F \leftarrow a \cdot d}$ you were completing the job of renaming the first number.

$$\frac{3}{8} \cdot \frac{4}{4} = \frac{\textcircled{12}}{32}$$

In $\boxed{S \leftarrow b \cdot c}$ you were renaming the second number.

$$\frac{3}{4} \cdot \frac{8}{8} = \frac{\textcircled{24}}{32}$$

Finally, you found the numerator of your answer: $\boxed{N \leftarrow F + S}$

so $$\frac{12 + 24}{32} = \frac{\textcircled{36}}{32}.$$

As you saw, the output was not the simplest form of the answer, because

$$\frac{36}{32} = \underline{\qquad\qquad}.$$

4. Can you get the simplest form of the answer by some other method? Look at the denominators of $\frac{3}{8}$ and $\frac{3}{4}$. 8 is a ____________ of 4, so the least common multiple of 8 and 4 is ________.

You can rename $\frac{3}{4}$ so that the new fraction has the denominator 8.

$$\frac{3}{4} = \frac{\quad}{8} \qquad \text{(write the numerator)}$$

Now you can add.

$$\frac{3}{8} + \frac{3}{4} = \frac{3}{8} + \frac{6}{8}$$

$$= \frac{\quad + \quad}{8}$$

$$= \frac{\quad}{8}$$

Is your answer in simplest form? ________

5. (a) Use the flow chart to find the answer.

$$\frac{2}{3} + \frac{1}{6} = \underline{\qquad\qquad}.$$

(b) Is the output, $\frac{N}{D}$, in simplest form? ________

(c) What is the least common multiple of 3 and 6 ? ________

Rename $\frac{2}{3}$:

$$\frac{2}{3} = \frac{\quad}{6}.$$

(d) Do the problem the usual way, using the new name for $\frac{2}{3}$.

$$\frac{4}{6} + \frac{1}{6} = \frac{\quad + \quad}{6}$$

$$= \frac{\quad}{6}$$

(e) Is your answer in simplest form? ________

6. (a) Use your flow chart to find the answer: $\frac{2}{3} + \frac{3}{4} =$ ________

(b) Is your answer in simplest form? ________

(c) What is the least common multiple of 3 and 4? ________

As you see, sometimes your flow chart gives you the answer in simplest form and sometimes it doesn't.

7. (a) Use the flow chart to add: $\frac{2}{3} + \frac{5}{6} =$ ____________

(b) Simplify your answer. ____________

(c) Since $\frac{2}{3} = \frac{4}{6}$, you can write:

$$\frac{2}{3} + \frac{5}{6} = \frac{4}{6} + \frac{5}{6}$$

$$= \frac{\quad + \quad}{6}$$

$$= \underline{\qquad\quad}$$

(d) Is this answer in simplest form? ____________

(e) Simplify it. ____________

You may not always get the answer in simplest form even when you use the least common denominator, but you are more likely to do so than when you use the flow chart.

Exercises

Use the flow chart or find the least common denominator to find the sums in the following problems. Write your answers in simplest form. You may want to use both methods a few times in order to check your answers.

1. $\frac{1}{2} + \frac{1}{3} =$ ____________
2. $\frac{1}{4} + \frac{4}{3} =$ ____________
3. $\frac{2}{5} + \frac{1}{4} =$ ____________
4. $\frac{1}{2} + \frac{5}{6} =$ ____________
5. $\frac{3}{4} + \frac{1}{2} =$ ____________
6. $\frac{5}{4} + \frac{2}{3} =$ ____________
7. $\frac{1}{8} + \frac{1}{16} =$ ____________
8. $\frac{5}{9} + \frac{^{-}1}{3} =$ ____________
9. $\frac{4}{5} + \frac{1}{8} =$ ____________
10. $\frac{5}{16} + \frac{3}{8} =$ ____________

11. $\frac{^-1}{12} + \frac{2}{3} =$ ________

12. $\frac{1}{3} + \frac{3}{4} =$ ________

13. $\frac{1}{2} + \frac{^-5}{12} =$ ________

14. $\frac{5}{8} + \frac{1}{2} =$ ________

15. $\frac{7}{16} + \frac{3}{4} =$ ________

16. $\frac{^-9}{8} + \frac{3}{4} =$ ________

17. $\frac{2}{5} + \frac{1}{10} =$ ________

18. $\frac{1}{6} + \frac{^-1}{5} =$ ________

19. $\frac{6}{5} + \frac{3}{10} =$ ________

20. $\frac{5}{2} + \frac{5}{4} =$ ________

Subtracting Rational Numbers

When you have a subtraction problem using rational numbers, you rewrite the problem as an addition problem and add the opposite of the subtrahend.

Example.

$$\begin{aligned} \frac{1}{2} - \frac{1}{4} &= \frac{1}{2} + \frac{{}^-1}{4} \\ &= \frac{2}{4} + \frac{{}^-1}{4} \\ &= \frac{2 + {}^-1}{4} \\ &= \frac{1}{4} \end{aligned}$$

So $\frac{1}{2} - \frac{1}{4} = \frac{1}{4}$

Exercises

Rewrite each subtraction problem as an addition problem and find the answer. You may use your flow chart if you wish.

1. $\frac{5}{8} - \frac{1}{2} =$ __________

   ____

2. $\frac{3}{4} - \frac{1}{3} =$ __________

   ____

3. $\frac{1}{2} - \frac{5}{6} =$ __________

   ____

4. $\frac{2}{3} - \frac{1}{6} =$ __________

   ____

5. $\frac{1}{8} - \frac{1}{2} =$ ____________

____

6. $\frac{1}{4} - \frac{1}{3} =$ ____________

____

7. $\frac{1}{8} - \frac{1}{16} =$ ____________

____

8. $\frac{5}{8} - \frac{1}{16} =$ ____________

____

9. $\frac{2}{3} - \frac{5}{6} =$ ____________

____

10. $\frac{5}{6} - \frac{2}{3} =$ ____________

____

11. $\frac{7}{8} - \frac{1}{2} =$ ____________

____

12. $\frac{7}{8} - \frac{3}{4} =$ ____________

____

Rational Numbers That Are Powers of Ten

In Chapter 4 you saw that 10^2 means 10×10, and 10^3 means $10 \times 10 \times 10$, or 1000. We can put this information in a chart.

10,000	1000	100	10	1			
10^4	10^3	10^2	10^1				

In the first row of boxes, each number is 10 times the one to the right of it.

$$\begin{aligned} 10 &= 10 \times 1 \\ 100 &= 10 \times 10 \\ 1000 &= 10 \times 100 \\ 10{,}000 &= 10 \times 1000 \end{aligned}$$

and so on.

As you do the exercises below you will fill in the other boxes of the chart.

Class Discussion

1. (a) 1 is 10 times what number? __________

   In the chart above, write $\frac{1}{10}$ in the box to the right of 1.

   (b) $\frac{1}{10} = 10 \times \underline{\ \ ?\ \ }$

   If you don't know, think of the way multiplication and division are related. That is:

   $$3 \times 8 = 24, \quad \text{so} \quad \frac{24}{3} = 8.$$

   In the problem, $\frac{1}{10} = 10 \times \underline{\ \ ?\ \ }$, you can find the answer by dividing $\frac{1}{10}$ by 10. Remember that when you divide you just multiply by the reciprocal of the divisor, so

   $$\begin{aligned} \frac{\frac{1}{10}}{10} &= \frac{1}{10} \times \frac{1}{10} \\ &= \underline{\qquad\qquad}. \end{aligned}$$

   Therefore, $\frac{1}{10} = 10 \times \underline{\qquad\qquad}$.

   Write $\frac{1}{100}$ in the box to the right of $\frac{1}{10}$.

(c) $\frac{1}{100} = 10 \times$ _____ .

Write $\frac{1}{1000}$ in the box to the right of $\frac{1}{100}$.

2. Now look at the second row of boxes. Each exponent is 1 greater than the exponent of the number to the right of it.

$$10^{\textcircled{4}} \quad 10^{\textcircled{3}} \quad 10^{\textcircled{2}} \quad 10^{\textcircled{1}}$$

Also, notice that the exponent shows how many zeros there are after the 1 in each number.

(a) You can write the other numbers from the top row as powers of ten also. What exponent do you think will be used for the number 1 ? __________

Mathematicians agree that $10^0 = 1$.

Write 10^0 in the box below 1 .

(b) 0 is 1 greater than what number? __________

We agree that 10^{-1} is another name for $\frac{1}{10}$. Write 10^{-1} in the box below $\frac{1}{10}$. Notice that the "1" part of the exponent shows how many zeros there are in $\frac{1}{10}$. The negative symbol shows that the 10 is below the bar in the fraction, $\frac{1}{10}$.

(c) What exponent will be used with 10 to show $\frac{1}{100}$? __________

Write 10^{-2} below $\frac{1}{100}$.

(d) What power of 10 is $\frac{1}{1000}$? __________

Write 10^{-3} below $\frac{1}{1000}$.

3. Using exponents makes work with powers of 10 very easy. You know that

$$\begin{aligned} 10^5 \times 10^3 &= 100{,}000 \times 1000 \\ &= 100{,}000{,}000 \;. \end{aligned}$$

You get this answer easily by adding the exponents 5 and 3 .

$$\begin{aligned} 10^5 \times 10^3 &= 10^{5+3} \\ &= 10 \end{aligned}$$

(Write the exponent.)

Division is also easy with powers of 10 . To divide 10^5 by 10^2 you can write $\frac{100000}{100} = 1000$ or you can just subtract the exponent of the divisor.

$$\begin{aligned}\frac{10^5}{10^2} &= 10^{5-2}\\ &= 10^{5+{}^{-}2}\\ &= 10 \qquad \text{(Write the exponent.)}\end{aligned}$$

This is especially helpful in problems like 100,000,000 divided by $\frac{1}{1000}$.

The "old" way

$$\begin{aligned}\frac{100,000,000}{\frac{1}{1000}} &= 100,000,000 \times 1000\\ &\quad \text{(multiply by the reciprocal of the divisor.)}\\ &= 100,000,000,000\end{aligned}$$

The "exponent" way

$$\begin{aligned}\frac{10^8}{10^{{}^{-}3}} &= 10^{8-{}^{-}3}\\ &= 10^{8+3}\\ &= 10\end{aligned}$$

(Write the exponent.)

Exercises

1. In the following problems, use the method of adding or subtracting exponents first. Then, at the right, check your answer by doing the problem without exponents.

Example. $10^2 \times 10^{{}^{-}1} = 10^{2+{}^{-}1}$ and $100 \times \frac{1}{10} = 10$

$= 10^1$

(Check: $10^1 = 10$.)

(a) $10^3 \times 10^2 =$ ________ and ________ × ________ = ________

= ________

(Check: ________ = ________.)

(b) $10^5 \times 10^{-3}$ = ________ and ________ × ________ = ________

= ________

(Check: ________ = ________.)

(c) $10^{-2} \times 10^4$ = ________ and ________ × ________ = ________

= ________

(Check: ________ = ________.)

(d) $10^{-3} \times 10^{-2}$ = ________ and ________ × ________ = ________

= ________

(Check: ________ = ________.)

(e) $\frac{10^5}{10^4}$ = ________ and 100,000 × $\frac{1}{10,000}$ = ________

= ________

(Check: ________ = ________.)

(f) $\frac{10^{-5}}{10^4}$ = ________ and ________ × ________ = ________

= ________

(Check: ________ = ________.)

2. Write these problems using the exponent form and write the answers with exponents.

(a) $\dfrac{10{,}000{,}000}{\frac{1}{100}}$ = ____________

= ____________

(b) $\dfrac{\frac{1}{100}}{\frac{1}{10{,}000}}$ = ____________

= ____________

(c) $\frac{1}{1{,}000} \times 10{,}000$ = ____________ × ____________

= ____________

(d) $\dfrac{\frac{1}{10{,}000}}{10}$ = ____________

= ____________

(e) $\dfrac{\frac{1}{100{,}000}}{\frac{1}{10}}$ = ____________

= ____________

Decimals and Powers of Ten

In Chapter 10 you learned how to rewrite fractions as decimal numerals. If a fraction has a denominator that is a power of 10 , there are just as many decimal places in the decimal numeral as there are zeros in the denominator.

$$\frac{1}{1,000} = .001 \quad \text{and} \quad .01 = \frac{1}{100}$$

Class Discussion

Since another name for $\frac{1}{10}$ is 10^{-1}, and $\frac{1}{10} = .1$, you can see that 10^{-1} is also a name for .1 .

Since $10^{-2} = \frac{1}{100}$, you can see that 10^{-2} is also a name for __________ (decimal numeral). The digit 2 in the exponent $^{-}2$ tells the number of decimal places in the decimal numeral.

Finish this chart.

Power of 10	Decimal Numeral	Fraction
10^6	1,000,000.	
10^5	______	
10^4	______	
10^3	______	
______	100.	
______	10.	
10^0	______	
10^{-1}	______	$\frac{1}{10}$
______	______	$\frac{1}{100}$
______	______	$\frac{1}{1,000}$
10^{-4}	______	______

(Of course you can write all the decimal numerals as fractions using a power of 10 as the denominator. For instance, $1 = \frac{10}{10}$ or $\frac{100}{100}$ or $\frac{1}{1}$.)

To multiply 100,000 by .001 , you can use the decimal numerals:

$$\begin{array}{r} 100{,}000 \\ \underline{.001} \\ 100.000 \end{array}$$

You can use the fraction form:

$$100{,}000 \times \frac{1}{1{,}000} = \frac{100{,}000}{1{,}000} = \underline{\qquad\qquad}$$

Or you can use the exponent form: $10^5 \times 10^{-3} = 10^{5+-3} = \underline{\qquad\qquad}$.

Using exponents is especially helpful when you have very small or very large numbers:

$$1{,}000{,}000 \times 1{,}000{,}000 = 1{,}000{,}000{,}000{,}000$$

or

$$10^6 \times 10^6 = \underline{\qquad\qquad}$$

$$.00001 \times .000001 = .00000000001$$

$$10^{-5} \times 10^{-6} = \underline{\qquad\qquad}$$

You can use exponents when you divide by a power of 10 written as a decimal numeral.

Decimal form:

$$\begin{aligned} \frac{100}{.0001} &= \frac{100}{.0001} \times \frac{10{,}000}{10{,}000} \\ &= \frac{1{,}000{,}000}{1} \\ &= 1{,}000{,}000 \end{aligned}$$

or exponent form:

$$\begin{aligned} \frac{10^2}{10^{-4}} &= 10^{2\ -\ {}^-4} \\ &= 10^{2\ +\ 4} \\ &= \underline{\qquad\qquad} \end{aligned}$$

Exercises

1. Rewrite each problem below using exponents with 10 . Write your answer with an exponent.

(a) $10{,}000 \times .0001$ = ____________ × ____________
= ____________

(b) $100 \times 100{,}000$ = ____________ × ____________
= ____________

(c) $.001 \times .01$ = ____________ × ____________
= ____________

(d) $.00001 \times .0000001$ = ____________ × ____________
= ____________

(e) $\frac{1}{100} \times 1000$ = ____________ × ____________
= ____________

(f) $\dfrac{\frac{1}{1000}}{\frac{1}{10{,}000}}$ = ____________ × ____________
= ____________

(g) $\frac{10}{.001}$ = ____________ × ____________
= ____________

(h) $\frac{.0001}{.01}$ = ____________ × ____________
= ____________

2. Rewrite each problem below using decimal numerals. Write the answer as a decimal numeral.

(a) $10^5 \times 10^3$ = ______________ × ______________

= ______________

(b) $10^{-2} \times 10^{-3}$ = ______________ × ______________

= ______________

(c) $10^6 \times 10^{-4}$ = ______________ × ______________

= ______________

(d) $10^0 \times 10^{-2}$ = ______________ × ______________

= ______________

Scientific Notation

Scientists often have to use very large or very small numbers. They need very large numbers when they talk about the distance to the stars and they need very small numbers when they talk about the size of an atom. Some of these numbers, when written down, require a great many zeros. Writing a great many zeros is not only a lot of trouble but it also can lead to mistakes. If you have to multiply a number with many zeros, it is easy to write one too many or one too few and get the wrong answer.

For example, light travels at a speed of about 186,000 *miles per second*! The distance of stars from the earth is measured in *light years*, or the distance light travels in a year. This is a very large number indeed.

To find out, in the usual way, what a light year is, we first multiply 186,000 by 60 to find out how far light travels in 1 *minute*:

$$\begin{array}{r} 186{,}000 \\ \times\ 60 \\ \hline 11{,}160{,}000 \end{array} \quad \text{(miles in 1 minute)}$$

We multiply 11,160,000 by 60 to find how far light travels in one *hour*:

$$\begin{array}{r} 11{,}160{,}000 \\ \times\ 60 \\ \hline 669{,}600{,}000 \end{array} \quad \text{(miles in 1 hour)}$$

We multiply 669,600,000 by 24 to find how far it travels in one *day*:

$$\begin{array}{r} 669{,}600{,}000 \\ \times\ 24 \\ \hline 2\ 678\ 400\ 000 \\ 13\ 392\ 000\ 000 \\ \hline 16{,}070{,}400{,}000 \end{array} \quad \text{(miles in 1 day)}$$

Last, we can multiply 16,070,400,000 by 365 to find how far light travels in one year:

$$\begin{array}{r} 16,070,400,000 \\ \times\ 365 \\ \hline 80\ 352\ 000\ 000 \\ 964\ 224\ 000\ 000 \\ 4\ 821\ 120\ 000\ 000 \\ \hline 5,865,696,000,000 \end{array} \quad \text{(miles in 1 year)}$$

Suppose you wanted to find the distance in miles of a star that was 492 light years away from the earth. Think how many zeros you would have to write in multiplying

$$5,865,696,000,000 \times 492 .$$

(We won't do this here!)

Because it is so clumsy to work with so many digits, scientists use powers of 10 to make their work easier.

Class Discussion

1. You can think of any number as 10 times some other number.

$35 = 10 \times 3.5$

$48.9 = 10 \times 4.89$

$56.4 = 10 \times$ ________

2. You can think of any number as 100 times another number.

$650 = 100 \times 6.5$

$785.7 = 100 \times 7.857$

$966.4 = 100 \times$ ________

3. You can think of any number as $\frac{1}{10}$ times another number.

$.459 = \frac{1}{10} \times 4.59$

$.1561 = \frac{1}{10} \times 1.561$

$.20723 = \frac{1}{10} \times$ ________

In scientific notation, every number is written so that there is only one digit to the left of the decimal point in the number to be multiplied by a power of 10. (Naturally, you wouldn't write 1,000,000 as 1×10^6 or 5 as 5×10^0. You'd just write 10^6 and 5.)

This is not as complicated as it sounds. Let's go back to the light year problem, and let scientific notation help us do the work.

To begin with, we write 186,000 as 1.86 × ___?___. To find out what power of 10 we need, we start at the right of 186,000 (at the decimal point) and count back until there is just one digit left. That is five places, so $186{,}000 = 1.86 \times 10^5$.

To multiply that number by 60, we write 60 as 6×10^1.

$$\begin{aligned} 186{,}000 \times 60 &= (1.86 \times 10^5) \times (6 \times 10^1) \\ &= (1.86 \times 6) \times (10^5 \times 10^1) \end{aligned}$$

because it doesn't matter in what order you multiply.

Then multiply 1.86 by 6:

$$\begin{array}{r} 1.86 \\ \times\ 6 \\ \hline 11.16 \end{array}$$

But 11.16 has two places to the left of the decimal point. We rewrite it as 1.116×10^1.

Our miles-in-one-minute number now is

$$(1.116 \times 10^1) \times (10^5 \times 10^1)$$

and that is 1.116×10^7 (miles in 1 minute).

Next we multiply that number by 60 to find how far light travels in one hour, but again we use 6×10^1 as the name for 60.

$$(1.116 \times 10^7) \times (6 \times 10^1) = (1.116 \times 6) \times (10^7 \times 10^1)$$

Then we multiply:

$$\begin{array}{r} 1.116 \\ \times\ 6 \\ \hline 6.696 \end{array}$$

and we know the miles-in-one-hour can be written

$$6.696 \times 10^8.$$

Next 6.696×10^8 is multiplied by 24 to find how far light travels in one day.

$$(6.696 \times 10^8) \times (2.4 \times 10^1) = (6.696 \times 2.4) \times (10^8 \times 10^1)$$

Multiply:

$$\begin{array}{r} 6.696 \\ \times\ 2.4 \\ \hline 2\ 6784 \\ 13\ 3920 \\ \hline 16.0704 \end{array}$$

16.0704 can be written 1.60704×10^1, so the miles-in-one-day can be written

$$(1.60704 \times 10^1) \times (10^8 \times 10^1) \quad \text{or} \quad 1.60704 \times 10^{10} .$$

Last, we multiply by 365, which is 3.65×10^2.

$$\begin{array}{r} 1.60704 \\ \times\ 3.65 \\ \hline 803520 \\ 9642240 \\ 4\ 8211200 \\ \hline 5.8656960 \end{array}$$

So a light year is

$$5.8656960 \times 10^{12} \quad \text{miles} .$$

Suppose we had found this number in the first place. How would we know what it looked like in the usual form? To change from scientific notation to our "regular" notation, we just move the decimal point twelve places to the right. We have to put in 5 zeros to do this, and we see:

$$5.8656960 \times 10^{12} = 5,865,696,000,000 .$$

Exercises

1. Use the flow chart on Page 13-10e to rewrite each number in scientific notation. Remember that every whole number can be written with a decimal point after the ones place.

Example 1. $3,450,600 = \underline{?}$

Step (a): 3. (See Boxes 3 and 4 in the flow chart.)

Step (b): 3.4506 (See Box 5 in the flow chart.)

Step (c): You moved the point 6 places to the left. (See Boxes 6 and 7.)

Step (d): Then

$$3,450,600 = 3.4506 \times 10^{6}$$ (From Box 9.)

Example 2. $.0134567 = \underline{?}$

Step (a): 1. (See Boxes 2 and 4.)

Step (b): 1.34567 (See Box 5.)

Step (c): You moved the decimal point 2 places to the right. (See Boxes 6 and 7.)

Step (d): Then

$$.0134567 = 1.34567 \times 10^{-2}$$ (From Box 8.)

(a) 4350 = ______________

(b) 2407.35 = ______________

(c) .0001968 = ______________

(d) .045735 = ______________

(e) 574821.6 = ______________

(f) 457,000,000 = ______________

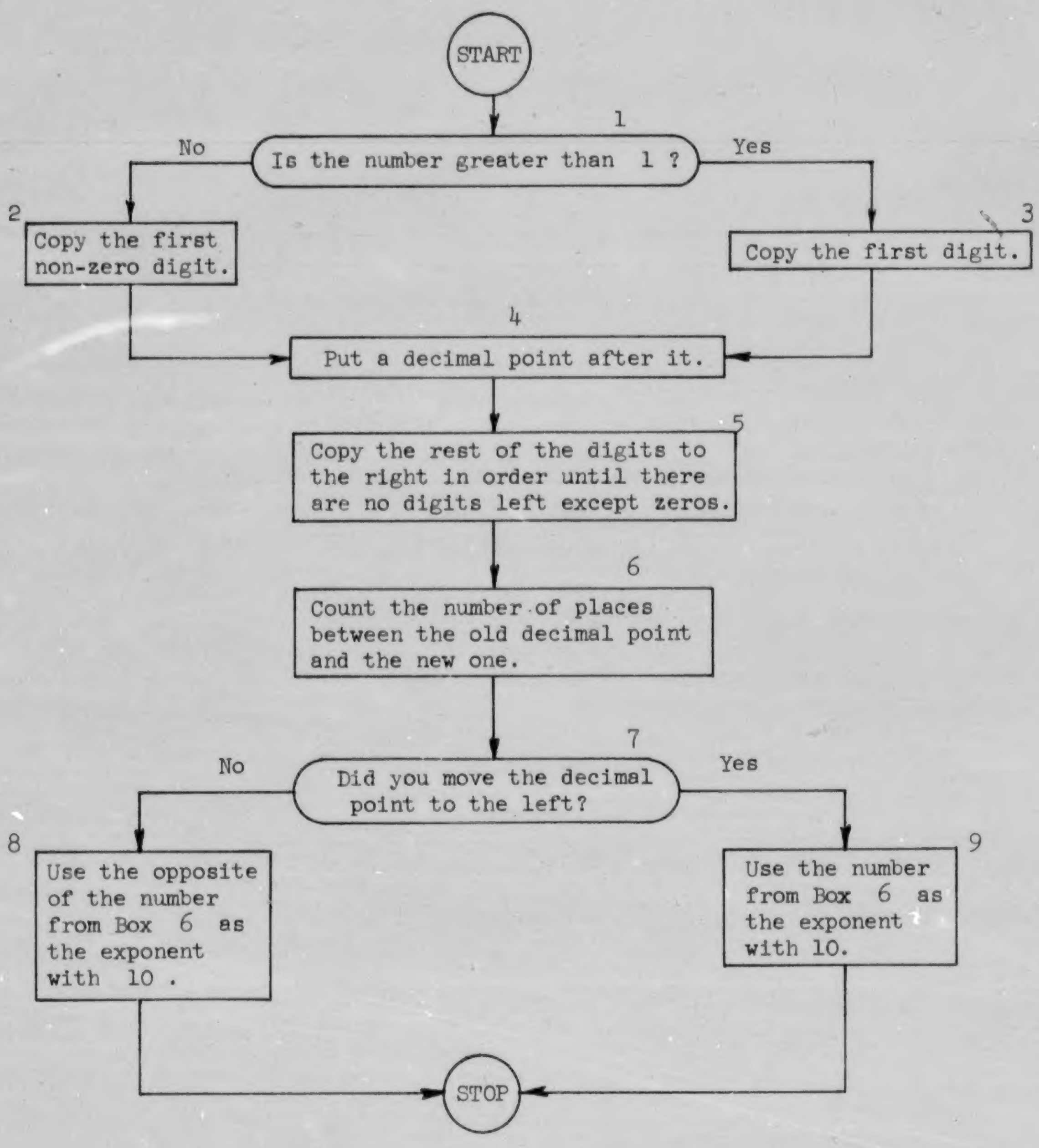
(g) .00846 = ______________

(h) 99,452,760 = ______________

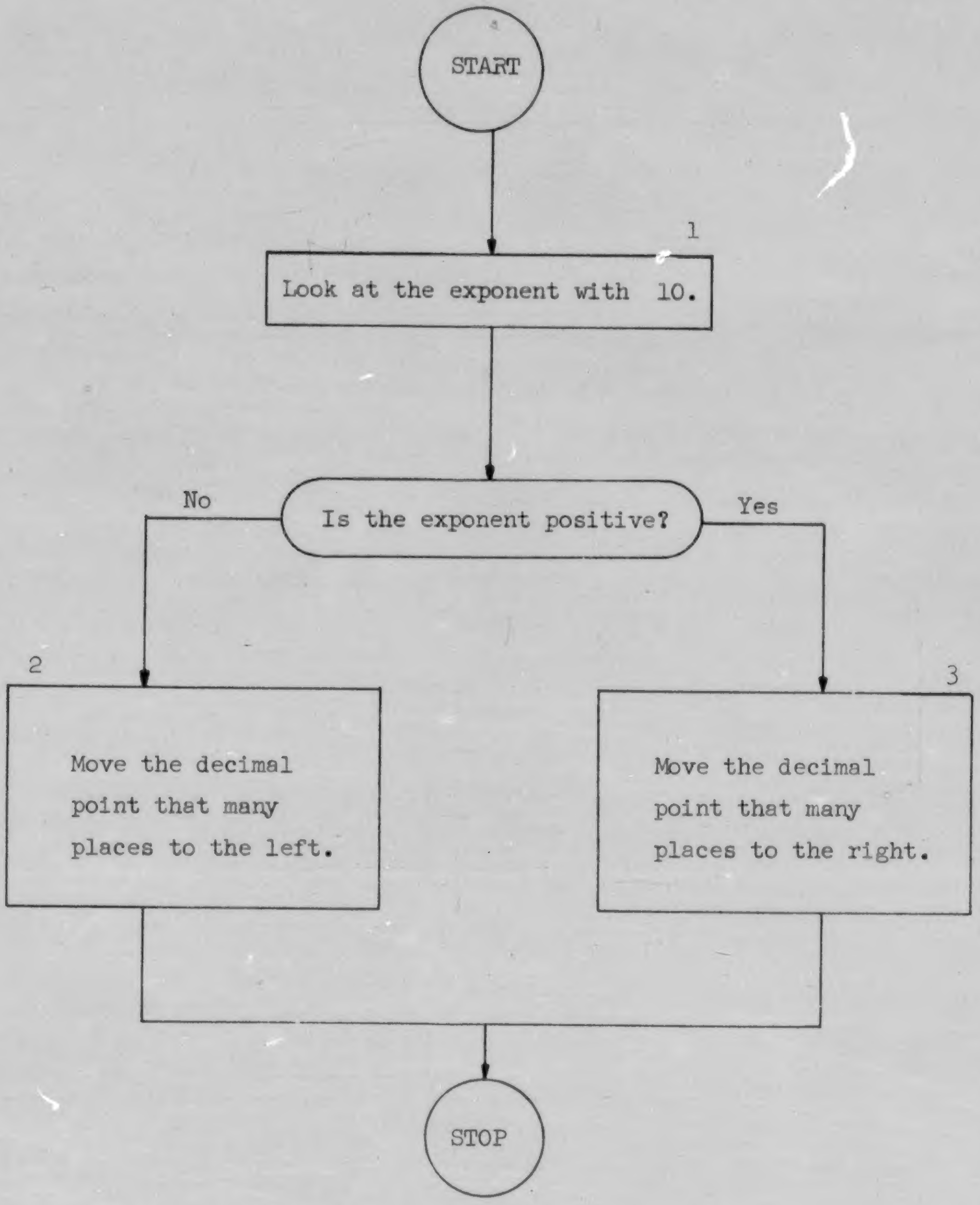
(i) 38.4093 = ______________

(j) .75369 = ______________

Flow Chart for Writing Scientific Notation


START
1
No
Is the number greater than 1 ?
Yes
2
Copy the first non-zero digit.
3
Copy the first digit.
4
Put a decimal point after it.
5
Copy the rest of the digits to the right in order until there are no digits left except zeros.
6
Count the number of places between the old decimal point and the new one.
7
No
Did you move the decimal point to the left?
Yes
8
Use the opposite of the number from Box 6 as the exponent with 10 .
9
Use the number from Box 6 as the exponent with 10.
STOP

Flow Chart for Changing Scientific Notation to Ordinary Numerals


START
1
Look at the exponent with 10.
No
Is the exponent positive?
Yes
2
Move the decimal point that many places to the left.
3
Move the decimal point that many places to the right.
STOP

2. These members are written in scientific notation. Use the flow chart on page 13-10f to write them the usual way.

Example 1. $3.45 \times 10^{-3} = \underline{\ ?\ }$

Step (a): $^{-}3$ (See Box 1.)

Step (b): $\underbrace{.003}\cancel{.}45$ (See Box 2.)

So $3.45 \times 10^{-3} = .00345$

Example 2. $2.019 \times 10^{5} = \underline{\ ?\ }$

Step (a): 5 (See Box 1.)

Step (b): $2\cancel{.}\underbrace{01900}.$ (See Box 3.)

So $2.019 \times 10^{5} = 201900$

(a) $6.15 \times 10^{-5} =$ ____________

(b) $1.001 \times 10^{3} =$ ____________

(c) $5.492 \times 10^{2} =$ ____________

(d) $2.8875 \times 10^{-4} =$ ____________

(e) $9.6264 \times 10^{6} =$ ____________

(f) $7.3485 \times 10^{-2} =$ ____________

Rational Numbers on the Number Line

Class Discussion

Take pages 13-11d and 13-11e out of your notebook.

The length of the unit segment used on each line is the same.

On line A , the unit is divided into hundredths. Find the point that corresponds to ten hundredths. Label it $\frac{10}{100}$. Then find the points that correspond to $\frac{20}{100}$, $\frac{30}{100}$, $\frac{40}{100}$, and so on, and label them.

On line B the unit is divided into hundredths again. Find the point that corresponds to $\frac{10}{100}$ on this line. Label that point 10% . Then find and label the points that correspond to 20% , 30% , 40% , and so on.

On line C, the unit is divided into ________ parts. Label the points $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, and so on.

On line D, the unit is again divided into ________ parts. Label the points .1 , .2 , .3 , and so on.

On line E, the unit is divided into ________ parts. Label the points $\frac{1}{8}$, $\frac{2}{8}$, $\frac{3}{8}$, and so on.

Label the points on lines F , G , and H to show what rational number each point corresponds to. Do not simplify the fractions. (That is, write $\frac{2}{4}$, $\frac{3}{6}$, etc. instead of $\frac{1}{2}$.)

Carefully cut along the dashed lines so that you have eight separate number lines. Keep line H apart for awhile, but stack the others up on your desk with line G on top.

Your teacher will give you three pins. Make sure the number lines are stacked up so that when you stick a pin through the point labeled 0 on line G it goes through the point labeled 0 on all the other lines, too. Leave the pin there.

Stick another pin through the stack so it goes through the point labeled 1 on every number line.

Stick the last pin through the point labeled $\frac{1}{4}$ so that it goes straight down through all the number lines.

1. Now carefully take out the pins.

On line E , the pin went through the point labeled ________

On line F , the pin went through a point between ________ and ________, and closer to ________.

On line D , the pin went through the point half-way between ________ and ________.

On line C , the pin went through the point half-way between ________ and ________.

On line B , the pin went through the point half-way between ________ and ________. This point corresponds to ________%.

On line A , the pin went through the point half-way between ________ and ________. Label this point as a fraction with the denominator 100 .

Stack up lines A through F again, with F on top. Stick pins through the points labeled 0 and 1 as before. Stick the third pin through the point labeled $\frac{3}{5}$. Take out the pins.

2. Show the point where the pin went through the other number lines.

A ________

B ________

C ________

D ________

E between ________ and ________, and closer to ________.

Stack up the number lines with A on the bottom in this order: A , B , C , D , G , F . Stick pins through points 0 and 1 . Stick the third pin through the point labeled $\frac{7}{8}$. Take out the pins.

3. Show where the pin went through the other number lines.

A halfway between $\frac{}{100}$ and $\frac{}{100}$ (Write the numerators.)

B between ________% and ________%.

C between ________ and ________ and closer to ________.

D between ________ and ________ and closer to ________.

G halfway between ________ and ________.

Stack up all the lines with H on top. Stick the third pin through the point labeled $\frac{1}{3}$. If you have been careful every time, the pin went through a point not marked on any of the other lines.

4. Show where the pin went through on each line.

A between ________ and ________

B between ________ and ________

C between ________ and ________

D between ________ and ________

E between ________ and ________

F between ________ and ________

G between ________ and ________

Since the unit segment is the same length on each number line, you can use this method to find different names for some other rational numbers, too.

Exercises

Think of a pin going through each point named. Show where it goes through each of the number lines given. (Use the lines and pins if you need them.)

1. $\frac{3}{4}$ E: ________ B: ________% A: ________

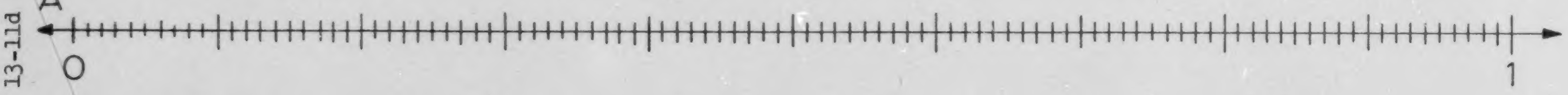
2. $\frac{2}{5}$ D: ________ C: ________ B: ________ A: ________

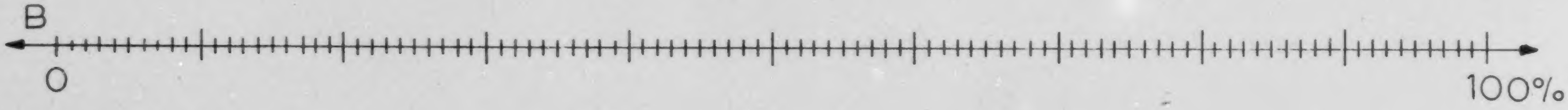
3. $\frac{4}{8}$ H: half-way between ________ and ________

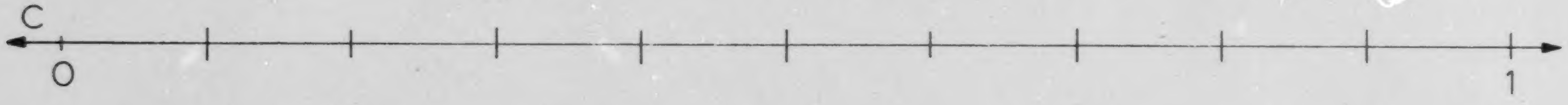
G: ________ C: ________ B: ________

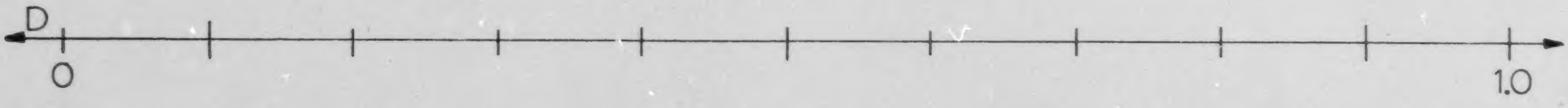
4. $\frac{7}{10}$ A: ________ B: ________ D: ________

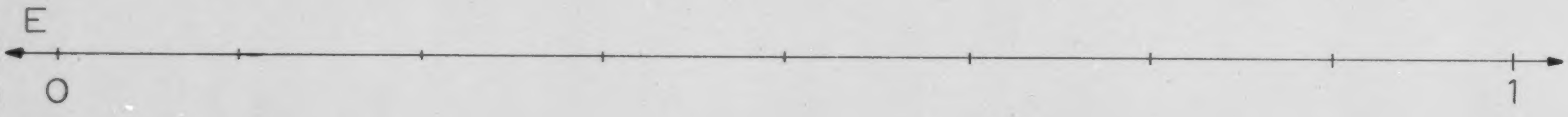
F: half-way between ________ and ________

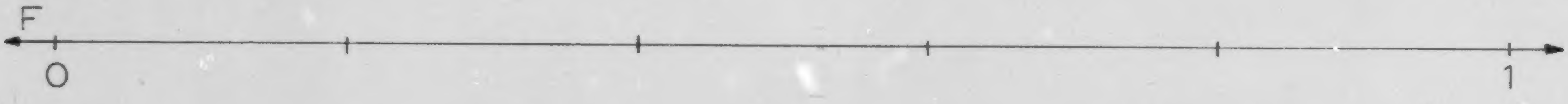

A
0
1

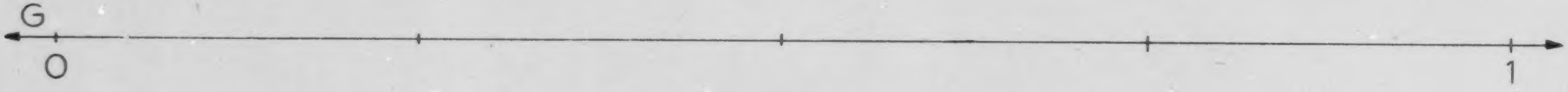

B
0
100%

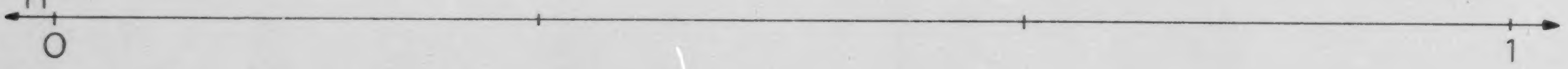

C
0
1


D
0
1.0


E
0
1


F
0
1


G
0
1


H
0
1

Fractions, Decimals and Percents

Suppose somebody offered you a choice like this:

(1) You can have $\frac{2}{5}$ of 400 dollars.

(2) You can have $.4 \times 400$ dollars.

(3) You can have 40% of 400 dollars.

Which would you choose?

You know it wouldn't make any difference because $\frac{2}{5}$, .4, and 40% all name the same number. You would get $160 no matter which you chose.

Any rational number can be written as a fraction, a decimal numeral, or a per cent. $\frac{2}{5} = .4$ and $\frac{2}{5} = 40\%$.

In Chapter 10 you learned how to rename fractions as decimals and decimals as fractions. In Chapter 12 you learned that whenever a ratio is written with 100 in the denominator, you are writing a percent. $\frac{50}{100} = 50\%$. Since you know that $\frac{50}{100} = \frac{1}{2}$, you know that $50\% = \frac{1}{2}$. And since $\frac{1}{2} = .5$, $50\% = .5$.

Class Discussion

1. To rewrite a percent as a fraction, the first step is to copy the number, leaving off the % sign. Use this as the numerator of a fraction. The denominator is ________.

$75\% = \frac{\quad}{100}$ (Write the numerator.)

$20\% = \frac{20}{\quad}$ (Write the denominator.)

62% = ____ (Write the fraction.)

The next step is to simplify the fraction if possible.

$\frac{75}{100} = \frac{25}{25} \times \underline{\qquad\quad}$ so $\frac{75}{100} = \underline{\qquad\quad}$. (Write the fraction.)

$\frac{20}{100} = \frac{20}{20} \times \underline{\qquad\quad}$ so $\frac{20}{100} = \underline{\qquad\quad}$.

$\frac{62}{100} = \frac{2}{2} \times \underline{\qquad\quad}$ so $\frac{62}{100} = \underline{\qquad\quad}$.

2. To rewrite a percent as a decimal, write the percent as a fraction with the denominator 100 .

$30\% = \frac{30}{\quad}$ (Write the denominator.)

$150\% = \frac{\quad}{100}$ (Write the numerator.)

$5\% = \underline{\qquad}$ (Write the fraction.)

Next, rewrite the fraction as a decimal numeral.

$\frac{30}{100} = \underline{\qquad\quad}$

$\frac{150}{100} = \underline{\qquad\quad}$

$\frac{5}{100} = \underline{\qquad\quad}$

3. Sometimes a percent has a fraction or a decimal in front of the % sign. Here are two examples: $1\frac{1}{2}\%$, $.5\%$. Again, write a fraction with the denominator 100 .

$$1\frac{1}{2}\% = \frac{1\frac{1}{2}}{100}$$

You know how to rewrite $1\frac{1}{2}$ as a decimal. $1\frac{1}{2} = \underline{\qquad\quad}$.

$$\text{So } \frac{1\frac{1}{2}}{100} = \frac{1.5}{100}$$

To divide by 100 , you move the decimal point ________ places to the ________, so 1.5 divided by 100 = ________. Therefore $1\frac{1}{2}\%$ = ________.

$.5\% = \frac{.5}{\quad}$ (Write the denominator.)

Divide .5 by 100 .

.5% = ________ (Write the decimal.)

4. Here are two ways to change a fraction to a percent. One way is to rename the fraction with the denominator 100 .

$\frac{1}{4} = \frac{\quad}{100}$ (Write the numerator.)

Rewrite the new fraction as a percent.

$\frac{25}{100}$ = ________%

For some fractions, the numerator of the new fraction may not be a whole number.

$\frac{5}{8} = \frac{?}{100}$

$5 \times 100 = 8 \times$ ________ (Divide 500 by 8 .) $8\overline{)500.0}$

So $\frac{5}{8} = \frac{62.5}{100}$ or $\frac{62\frac{1}{2}}{100}$

$\frac{5}{8} = 62.5\%$ or $62\frac{1}{2}\%$

Sometimes when you rename a fraction you get a repeating decimal. This brings us to the second way to change a fraction to a percent, and this is a way you can use with all fractions.

$\frac{1}{3} = ?\ \%$

You know that $\frac{1}{3}$ is written $.\overline{3}$ as a decimal. Since percent means hundredths, you can divide the numerator of the fraction by the denominator, but use only two places (tenths and hundredths place) to the right of the decimal point.

To find $\frac{1}{3}$, divide: $3\overline{)1.00}$ with quotient $.33\frac{1}{3}$.

So $\frac{1}{3} = .33\frac{1}{3}$ or $\frac{33\frac{1}{3}}{100}$ or $33\frac{1}{3}\%$.

5. To change a decimal to a percent, write the decimal as a fraction with a power of ten as the denominator.

$.875 = \frac{875}{}$ (Write the denominator.)

Now rename this fraction so that the denominator is 100.

$\frac{875}{1000} = \frac{?}{100}$ (To find the numerator, think $875 \times 100 = 1000 \times$ ______.)

$\frac{87.5}{100} = 87.5\%$

So $.875 = 87.5\%$

Remember that when you rename a decimal as a percent, you move the decimal point two places to the right and put a percent sign: $.375 = \cancel{.}37.5\%$ and $.2 = \cancel{.}20.\%$.

Exercises

1. Fill in the chart below so that each line gives three different names for the same number.

Fraction	Decimal	Percent
$\frac{1}{2}$	.5	50%
______	.25	______
$\frac{1}{8}$	______	______
______	.375	______
______	______	65%
$\frac{4}{5}$	______	______
______	.63	______
$\frac{2}{3}$	______	______
______	______	$12\frac{1}{2}$ %
______	______	130%
______	.45	______
______	______	$\frac{1}{4}$ %
______	______	5.5%
______	.001	______

2. Solve these problems. Use the way that's easiest for you. (Remember that "of" means "times".)

Example. $12\frac{1}{2}\%$ of $128 = ?$

One way: $\frac{12.5}{100} \times 128 = \frac{1600.0}{100}$

$= 16$

Another way: $.125 \times 128 = 16.000$

$= 16$

A third way: $\frac{1}{8} \times 128 = \frac{128}{8}$

$= 16$

(a) $\frac{1}{4}$ of $60 =$ ________

(b) $\frac{3}{8} + \frac{1}{5} =$ ________

(c) 25% of $480 =$ ________

(d) $.375 \times 72 =$ ________

(e) 60% of $100 =$ ________

(f) $66\frac{2}{3}\%$ of $150 =$ ________

(g) $.25 \times .875 =$ ________

(h) $.39 \times 200 =$ ________

(i) $\dfrac{276}{\frac{1}{2}} =$ ________

(j) $55 = 50\%$ of ________

(k) $\dfrac{3}{.125} =$ ________

Pre-Test Exercises

These exercises are like the problems you will have on the chapter test. If you don't know how to do them, read the section again. If you still don't understand, ask your teacher.

1. (Section 13-1.)

Find the answer to these addition and subtraction problems. Use the line below each subtraction problem to rewrite it as an addition problem.

(a) $4 + 6 =$ ________

(b) $13 - 4 =$ ________

____

(c) $10 + {}^-8 =$ ________

(d) ${}^-5 + 3 =$ ________

(e) ${}^-2 - 7 =$ ________

____

(f) $3 - 6 =$ ________

____

(g) ${}^-2 + 5 =$ ________

(h) $4 - {}^-2 =$ ________

____

(i) ${}^-6 - {}^-9 =$ ________

____

(j) ${}^-7 - {}^-3 =$ ________

____

(k) ${}^-7 + {}^-8 =$ ________

(l) ${}^-15 + 5 =$ ________

(m) $479 + {}^-479 =$ ________

2. (Section 13-2.)

Multiply or divide as shown.

(a) $4 \cdot 6 =$ ________

(b) $7 \cdot {}^-9 =$ ________

(c) $\frac{14}{2} =$ ________

(d) $\frac{15}{{}^-3} =$ ________

(e) ${}^-8 \cdot 4 =$ ________

(f) ${}^-5 \cdot {}^-5 =$ ________

(g) $\frac{{}^-13}{13} =$ ________

(h) $\frac{{}^-6}{2} =$ ________

3. (Section 13-3.)

Rewrite the numbers below if necessary. Then compare them. (Use < or >).

(a) $\frac{^-17}{3}$ ________ $\frac{14}{2}$

(b) $\frac{2}{^-6}$ ________ $\frac{^-5}{3}$

(c) $\frac{^-1}{^-4}$ ________ $\frac{2}{7}$

4. (Section 13-3.)

Rewrite if necessary and add.

(a) $\frac{1}{^-6} + \frac{3}{6} =$ ________

(b) $\frac{3}{4} + \frac{^-1}{^-4} =$ ________

5. (Section 13-4.)

Fill the blanks.

(a) The number __________ is a multiple of every number.

(b) Every number is a multiple of ________ .

(c) The first multiple of every number is the ____________________ ____________________.

(d) Every number is a __________________ of itself.

(e) The first five multiples of 4 (except 0, of course) are _____, _____, _____, _____, and _____.

(f) The multiples of 2 are called ____________ numbers.

(g) The first five prime numbers are _____, _____, _____, _____, and _____.

(h) Prime numbers are multiples of exactly __________ numbers.

6. (Section 13-5.)

Fill the blanks.

(a) If, in a pair of numbers, one of them is a multiple of the other, then the __________ number is the least common multiple of the pair.

(b) The least common multiple of 4 and 16 is _______.

(c) The least common multiple of two prime numbers is the __________ of the two.

(d) The L.C.M. of 5 and 7 is _______.

(e) If two numbers have no common factors except 1, their L.C.M. is the _____________ of the two numbers.

(f) The L.C.M. of 9 and 4 is _________.

(g) The L.C.M. of 15 and 18 is _________.

(h) Every common multiple of 2 and 3 is also a multiple of _________.

7. (Section 13-6.)

Add the following rational numbers. (You may use the flow chart on Page 7 of your tables if you need it.) Use the space at the right for your work.

(a) $\frac{1}{2} + \frac{1}{3} =$ _________

(b) $\frac{2}{3} + \frac{3}{4} =$ _________

(c) $\frac{1}{4} + \frac{3}{8} =$ _________

(d) $\frac{{}^{-}1}{4} + \frac{1}{9} =$ _________

(e) $\frac{3}{4} + \frac{{}^{-}5}{6} =$ _________

8. (Section 13-7.)

Rewrite each subtraction problem as an addition problem and then find the answer. Use the space at the right for your work.

(a) $\frac{3}{8} - \frac{1}{4}$ = ________

____

(b) $\frac{3}{4} - \frac{2}{3}$ = ________

____

(c) $\frac{9}{8} - \frac{5}{6}$ = ________

____

(d) $\frac{4}{5} - \frac{1}{2}$ = ________

____

(e) $\frac{^-1}{3} - \frac{1}{6}$ = ________

____

9. (Section 13-8.)

Use exponents to multiply or divide.

(a) $10^{^-1} \times 10^{^-4}$ = ________

= ________

(b) $\frac{10^5}{10^3}$ = ________

= ________

(c) $10^5 \times 10^{^-3}$ = ________

= ________

(d) $\frac{10^4}{10^{-1}}$ = ________

= ________

(e) $\frac{\frac{1}{1,000}}{\frac{1}{100}}$ = ________

= ________

(f) $\frac{1}{10,000} \times \frac{1}{10,000}$ = ________

= ________

10. (Section 13-9.)

Rewrite each problem using 10 and an exponent. Write your answer with an exponent.

(a) 1,000,000 × .01 = ________ × ________

= ______

(b) .0001 × .01 = ________ × ________

= ______

(c) .001 × 10,000 = ________ × ________

= ______

(d) $\frac{100,000}{.0001}$ = ______

____

= ____________

11. (Section 13-10.)

Write these numbers using scientific notation. Use the flow chart on page 13-10e if you need it.

(a) 456,000,000 = ________ × ________

(b) 297.457 = ________ × ________

(c) .0001642 = ________ × ________

(d) 54600.7 = ________ × ________

(e) .0087523 = ________ × ________

12. (Section 13-11.)

Fill in the chart.

Fraction	Decimal Numeral	Percent
$\frac{1}{2}$	.5	50%
________	________	87.5%
$\frac{1}{6}$	________	________
________	.9	________
________	________	$\frac{1}{2}$ %

TEST

1. Addition and Subtraction. Use the line below the problem to rewrite it if necessary.

(a) $5 + {}^{-}11 =$ ________

____

(b) $4 - 7 =$ ________

____

(c) ${}^{-}8 + 16 =$ ________

____

(d) $3 - {}^{-}4 =$ ________

____

(e) ${}^{-}9 - 2 =$ ________

____

2. Multiply.

(a) $3 \cdot 9 =$ ________

(b) ${}^{-}15 \cdot {}^{-}2 =$ ________

(c) ${}^{-}8 \cdot 3 =$ ________

(d) $4 \cdot {}^{-}7 =$ ________

3. Divide.

(a) $\frac{36}{{}^{-}4} =$ ________

(b) $\frac{{}^{-}9}{{}^{-}3} =$ ________

(c) $\frac{{}^{-}8}{8} =$ ________

4. Find the L.C.M. of each pair of numbers.

(a) 3 and 12: L.C.M. is __________

(b) 15 and 8: L.C.M. is __________

(c) 9 and 15: L.C.M. is __________

5. Add. Use the space at the right for your work.

(a) $\frac{2}{3} + \frac{1}{4} =$ __________

(b) $\frac{3}{5} + \frac{^-1}{10} =$ __________

(c) $\frac{9}{8} + \frac{1}{6} =$ __________

6. Use the line below each subtraction problem to rewrite it as an addition problem. Then find the answer. The space at the right is for your work.

(a) $\frac{3}{4} - \frac{1}{3} =$ __________

____

(b) $\frac{5}{6} - \frac{1}{3} =$ __________

____

(c) $\frac{5}{8} - \frac{3}{4} =$ __________

____

7. Rewrite the following using exponents. Use an exponent in your answer.

(a) $1000 \times 1000 = \_\_\_\_\_ \times \_\_\_\_\_$

$= \_\_\_\_\_$

(b) $.001 \times 1000 = \_\_\_\_\_ \times \_\_\_\_\_$

$= \_\_\_\_\_$

(c) $\frac{1000}{1,000,000} = \_\_\_\_\_\_\_\_\_\_$

$= \_\_\_\_\_\_\_$

(d) $\frac{\frac{1}{10}}{\frac{1}{1000}} = \_\_\_\_\_\_\_\_\_\_$

$= \_\_\_\_\_\_\_$

8. Write these numbers using scientific notation.

(a) 40.6 = __________

(b) 274,820,000 = __________

(c) .00006821 = __________

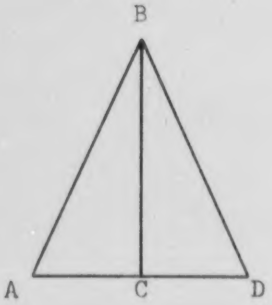
(d) .00567 = __________

9. Fill in the chart.

Fraction	Decimal Numeral	Percent
_____	_____	50%
$\frac{2}{3}$	_____	_____
_____	1.125	_____
_____	_____	1%
_____	.25	_____
$\frac{1}{1,000}$	_____	_____

Check Your Memory: Self-Test

1. (Sections 9-5 and 9-8.)

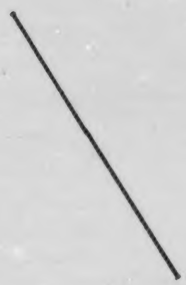


$\overline{BC}$ is the perpendicular bisector of $\overline{AD}$. Therefore,

$\triangle ABC \cong \triangle$ ________

by the __________ property of congruence.

2. (Section 9-4.)
Using the line segment below as one side, construct a triangle with all three sides congruent.



3. (Section 9-7.)
Use the triangle you just drew as one half of a rhombus and complete the rhombus.

4. (Section 10-6.)

Compare these numbers. Show which is smaller by putting < or > between them.

(a) .06 ______ .0395

(b) .2001 ______ .096

(c) 3.014 ______ 2.6

(d) .397 ______ .3971

5. (Section 10-13.)

Add or subtract. Watch the signs.

(a) .0321 + .76 = ________

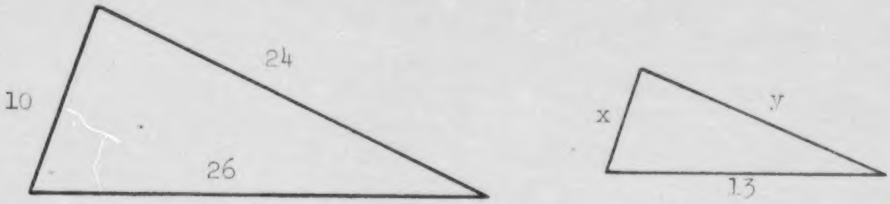
(b) 2.751 + 13.528 = ________

(c) .309 - .27 = ________

(d) .0126 - .00451 = ________

6. (Section 12-2.)

For the following pair of similar triangles, write the scale factors and then find the lengths that are missing.



Scale factor: ______ and ______

x = ________

y = ________

7. (Section 12-8.)

Write the missing numerators and denominators. Then solve the problem.

(a) To find 40% of 40 , you write:

$$\frac{\quad}{\quad} = \frac{x}{40}$$

(Use the space below for the arithmetic.)

40% of 40 is __________.

(b) To find what percent of 60 the number 45 is, you write:

$$\frac{x}{100} = \frac{\quad}{60}$$

_______% of 60 is 45.

(c) To find what number 18 is 50% of, you write:

$$\frac{\quad}{x} = \frac{50}{\quad}$$

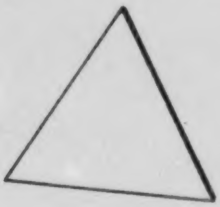
18 is 50% of ________.

Now check your answers on the next page. If you do not have them all right, go back and read the section again.

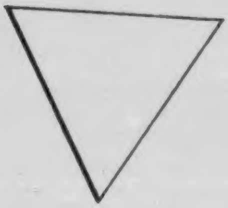
Answers to Check Your Memory: Self-Test

1. △ ABC ≅ △ DBC by the SAS congruence property.

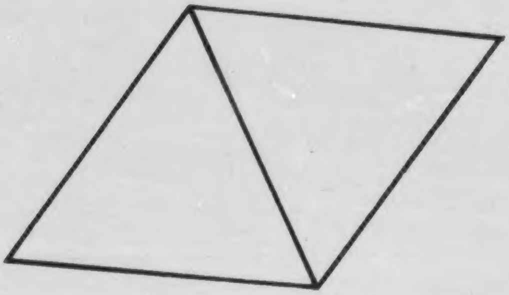
2. Your triangle should have looked like this:



or like this:



3. Your rhombus should be congruent to this one, but it may be turned around.



4. (a) $.06 > .0395$

   (b) $.2001 > .096$

   (c) $3.014 > 2.6$

   (d) $.397 < .3971$

5. (a) .7921

   (b) 16.279

   (c) .039

   (d) .00809

6. Scale factor: $\frac{1}{2}$ and 2 .

   $x = 5$
   $y = 12$

7. (a) $\frac{40}{100} = \frac{x}{40}$

   40% of 40 is $\underline{\ 16\ }$.

   (b) $\frac{x}{100} = \frac{45}{60}$

   $\underline{\ 75\%\ }$ of 60 is 45 .

   (c) $\frac{18}{x} = \frac{50}{100}$

   18 is 50% of $\underline{\ 36\ }$.

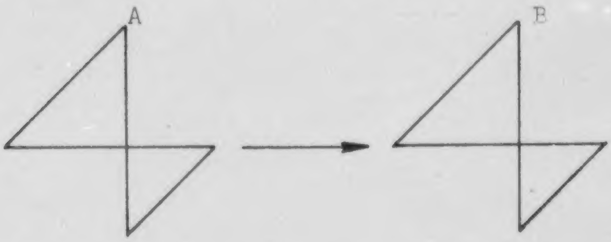
Chapter 14

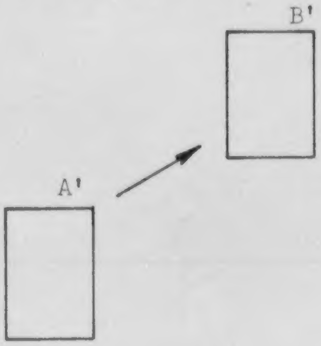
PERPENDICULARS

Chapter 14
PERPENDICULARS

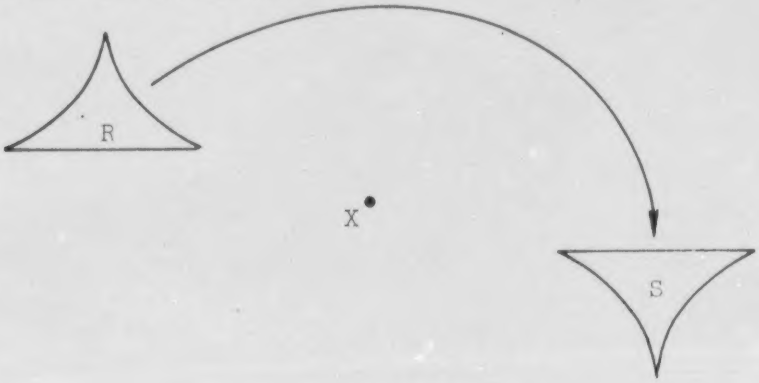
Congruent Figures and Motions

You have learned that two geometric figures are congruent to each other if they have exactly the same size and shape. Two figures that are congruent may be made to fit together by a motion. In this part of the chapter we will look at three types of motion. For example, the figure A on the left may be made to fit onto the figure B by a sliding motion in the direction of the arrow. We say that figure B is the slide image of figure A , or that the slide maps figure A onto figure B . Likewise, figure B' is a slide image of figure A' because a sliding motion in the direction of the right-hand arrow carries A' onto B' .



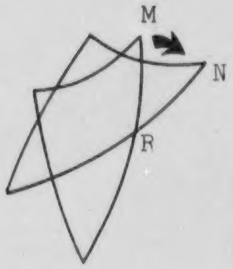


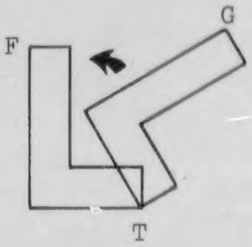
Look at the congruent figures shown below. There is no slide that will map figure R onto figure S .

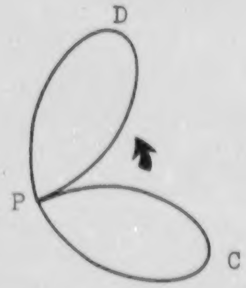


But figure R may be made to map onto figure S by a turn about point X as a center of turn. We say that figure S is the turn image of figure R , or that the turn maps figure R onto figure S .

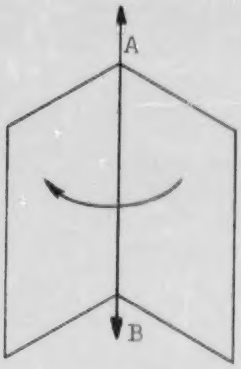
The center of turn may even be a point common to the two figures. Look at the congruent figures shown below.







The third basic motion that we will consider is the flip. You can think of a flip as the motion involved when you turn over a sheet of paper. In the process of flipping a sheet of paper the plane of the paper is rotated about a line as shown below.



In this case, the line $\overleftrightarrow{AB}$ is called the flip axis. Look at the congruent pair of figures, E and F, shown below.

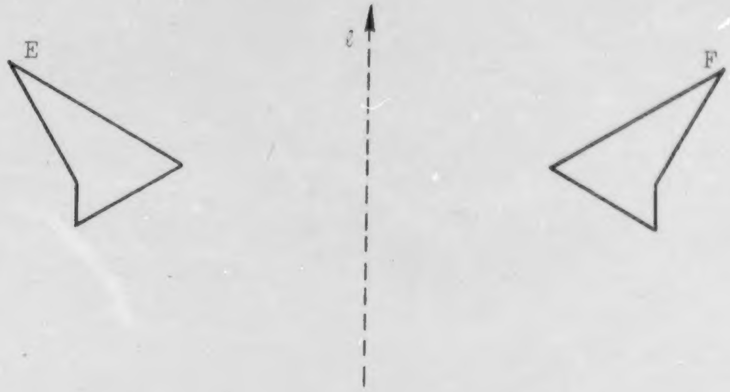
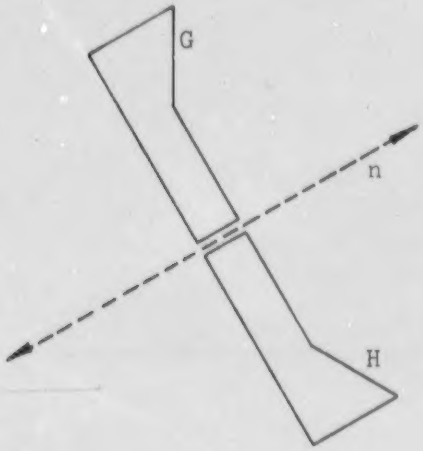
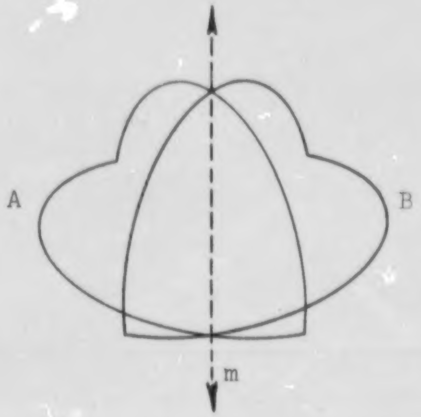


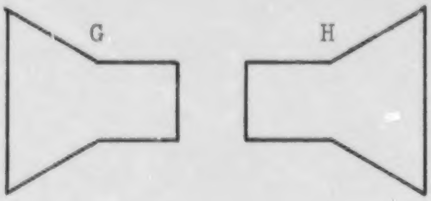
Figure E may be made to map onto figure F by a flip about ℓ as the flip axis. We say that figure F is the flip image of figure E or that the flip maps figure E onto figure F .

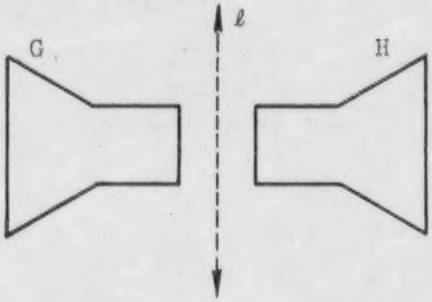
In the figure below, note how the figures are related by flips about n and m .



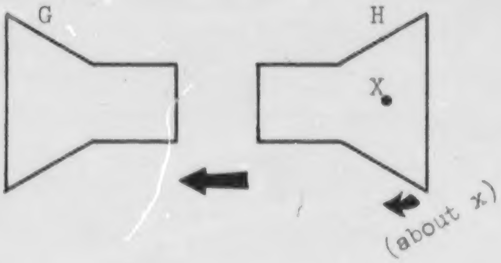


It is interesting to see that sometimes two congruent figures can be made to map onto each other by different motions. For example, figure H can be made to map onto figure G by a flip or by a turn/slide or by other ways. Find one other way.





flip

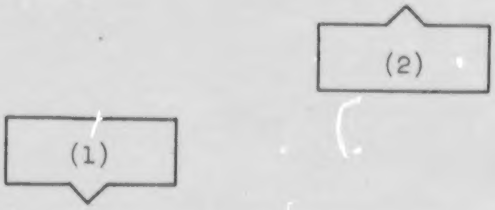


turn/slide

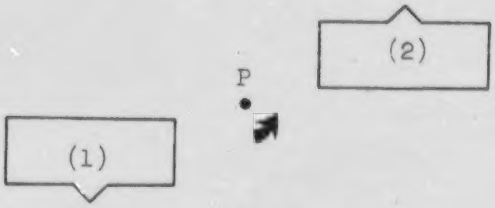
Exercises

1. In each case, the given pair of figures is congruent. For each congruent pair, show exactly how the figure marked (1) may be mapped onto the figure marked (2) by using the motions of slide, turn, or flip, or a combination of these methods. Use arrows to show slides or turns and dotted lines to show flip axes. Try to show more than one method of making the figures map onto each other.

Example.

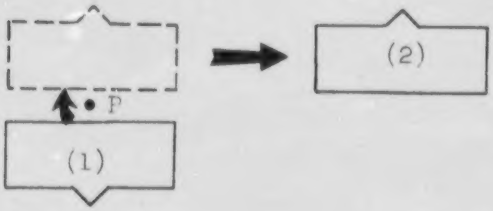


Possible Solution:



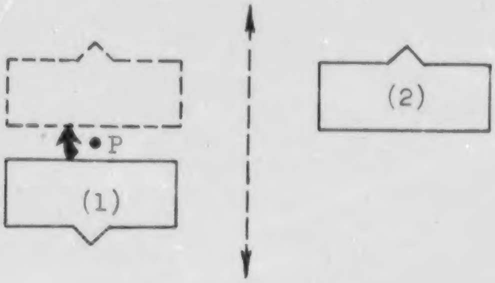
turn about Point P

Possible Solution:



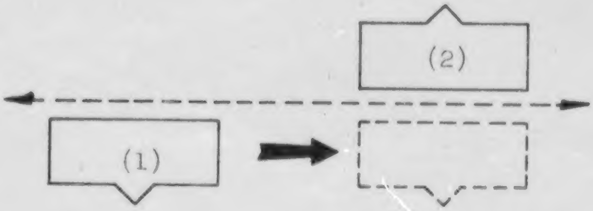
turn/slide

Possible Solution:



turn/flip

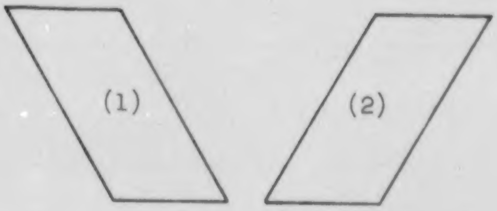
Possible Solution:



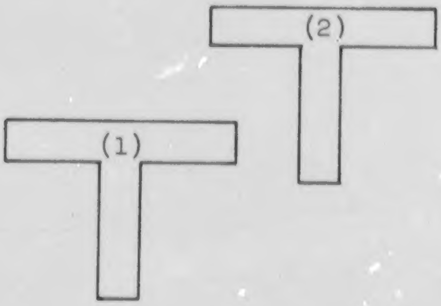
slide/flip

Can you find any others? (Hint: There are some!)

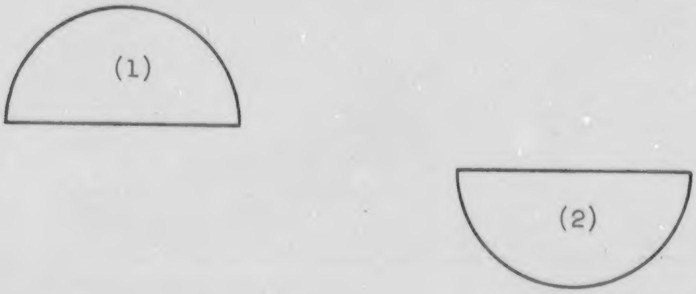
(a)



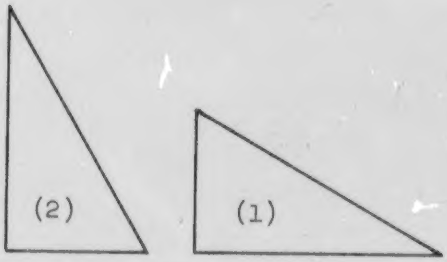
(b)



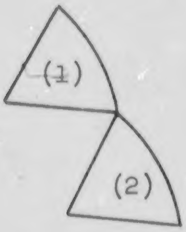
(c)



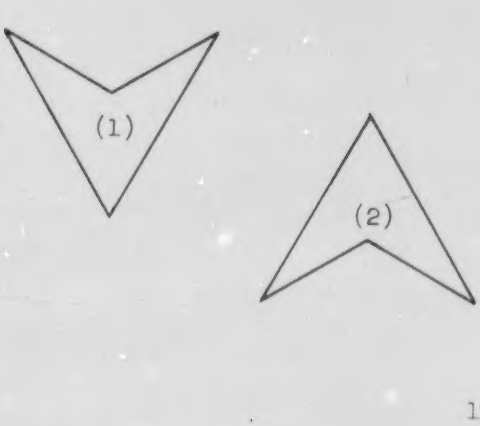
(d)



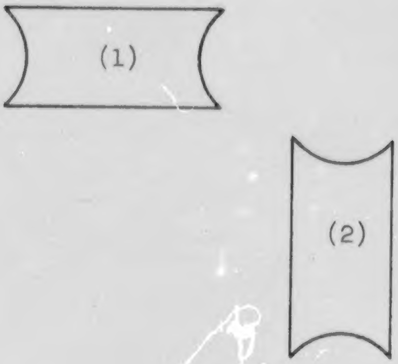
(e)



(f)

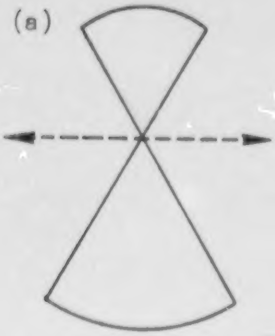


(g)

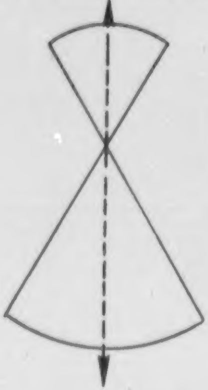


2. The figures below are to be flipped on the dotted lines as axes. In which cases does the figure map onto itself?

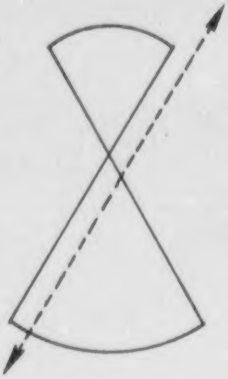
(a)



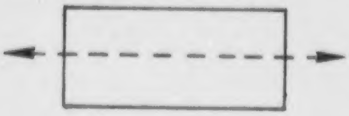
(b)



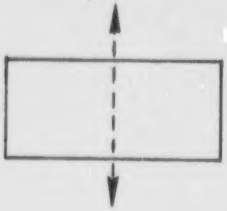
(c)



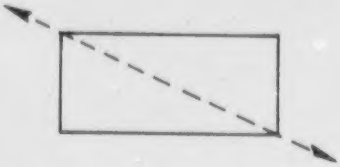
(d)



(e)

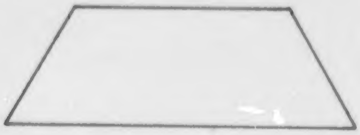


(f)

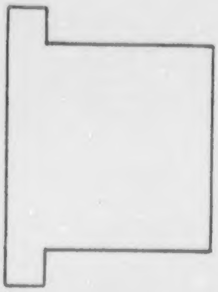


3. For each figure below, draw all flip axes such that the figure maps onto itself.

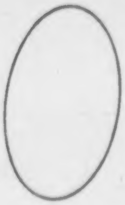
(a)



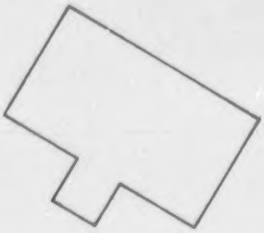
(b)



(c)



(d)

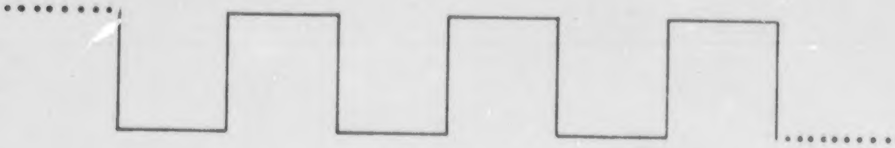


4. Can you think of a figure that has infinitely many flip axes?

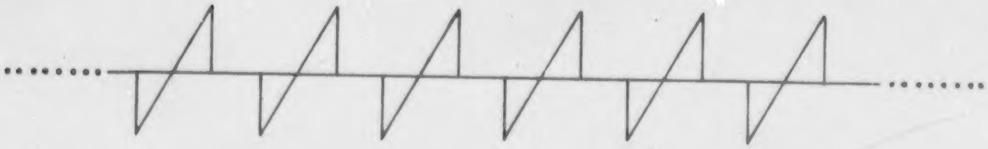
BRAINBOOSTER.

5. Think of each figure below as a wall design that extends forever in both directions. What motion or motions will carry the design onto itself?

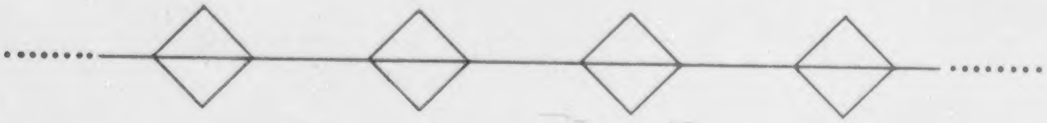
(a)



(b) Name two types of motion that will carry this design onto itself.

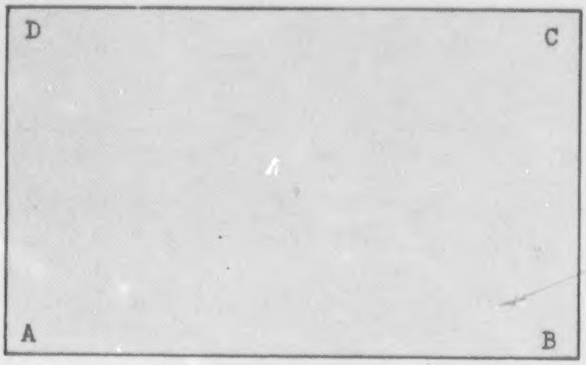


(c) Name two types of motion that will carry this design onto itself. Can you find a third type of motion?

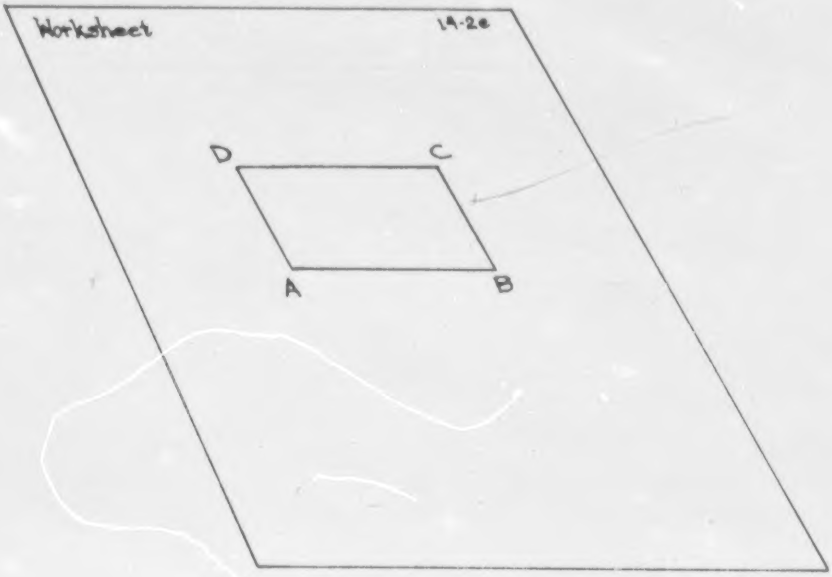


Congruence of a Figure with Itself

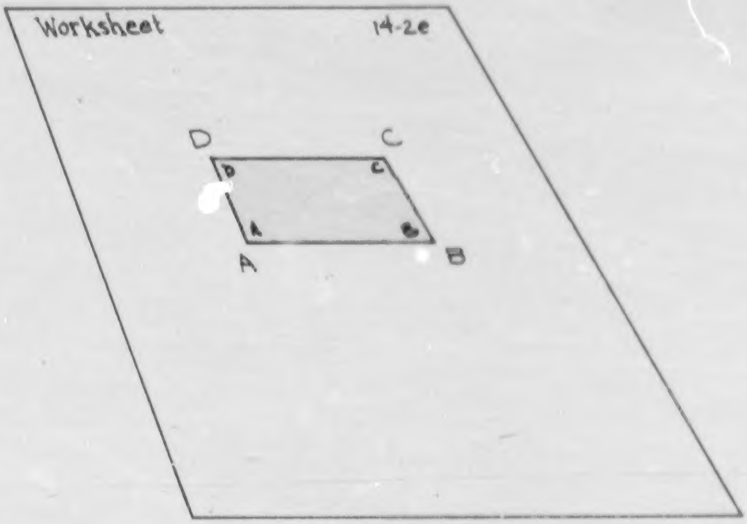
Before you start this lesson your teacher will give you a 3 × 5 card. Label the corners of this rectangular card with capital letters, as shown below. Now turn the card over and label the corners on the back so that A is in back of A , B is in back of B , C is in back of C , and D is in back of D .



Remove Page 14-2e from your notebook and place the card in the center of the page. Trace along the edges of the card so that you draw a rectangle. Label the vertices of the rectangle as shown below.



Now, in how many ways can you fit the card back onto the rectangle? One way would be to slide the card back onto the rectangle in the exact position it was when you made the tracing.



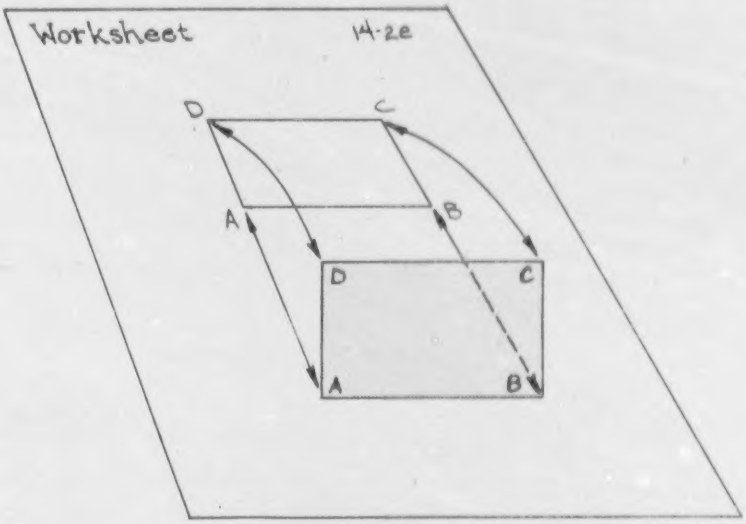
This fitting shows the identity congruence

$$ABCD \cong ABCD\ .$$

Notice that there is a correspondence between each corner of the card and the vertices of the rectangle that looks like this:

$$ABCD \cong ABCD\ .$$

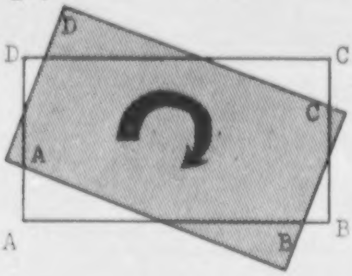
We always write congruences so that we can tell, even without looking at the picture, which parts of the figures correspond. The picture below is an example of what we mean by this correspondence.



Class Discussion

1. (a) Place your card back on the tracing so as to show the identity congruence, ABCD ≅ ABCD .

   (b) Now make a half-turn with the card so that the corner A takes a position at C , the corner B takes a position at D , the corner C takes a position at A , and the corner D takes a position at B .

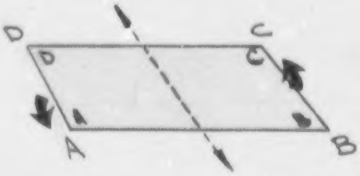


   (c) This fitting shows the congruence

   ABCD ≅ __________ .

2. (a) Place your card back on the tracing so as to show the identity congruence, ABCD ≅ ABCD .

   (b) Now take the card and flip it about its vertical axis



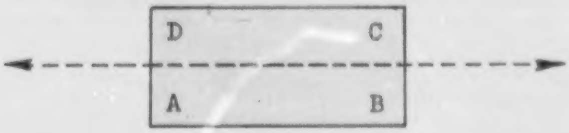
   so that corner C takes a position at D , corner B takes a position at A , corner D takes a position at C , and corner A takes a position at B .

   (c) This fitting shows the congruence

   ABCD ≅ __________.

3. (a) Place your card back on the tracing so as to show the identity congruence, ABCD ≅ ABCD .

(b) Flip your card about its horizontal axis



and fit it back on the tracing.

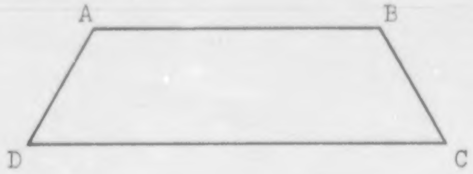
(c) This fitting shows the congruence

ABCD ≅ __________.

You can see that with a rectangle there are four different positions in which we can show congruence, and in each case the correspondences are different.

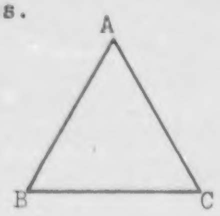
Exercises

1. The figure below is called an isosceles trapezoid with $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \cong \overline{BC}$. Write two correspondences that show that ABCD is congruent with itself.



(a) ABCD ≅ __________ (b) ABCD ≅ __________

2. The triangle below is an equilateral triangle. In an equilateral triangle all three sides are congruent. Write the correspondences that show that $\triangle ABC$ is congruent to itself. There are six such correspondences.

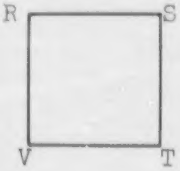


(a) $\triangle ABC \cong$ ________ (d) $\triangle ABC \cong$ ________

(b) $\triangle ABC \cong$ ________ (e) $\triangle ABC \cong$ ________

(c) $\triangle ABC \cong$ ________ (f) $\triangle ABC \cong$ ________

3. The figure below is a square. Write eight correspondences that show that RSTV is congruent to itself.



(a) RSTV $\cong$ ________ (e) RSTV $\cong$ ________

(b) RSTV $\cong$ ________ (f) RSTV $\cong$ ________

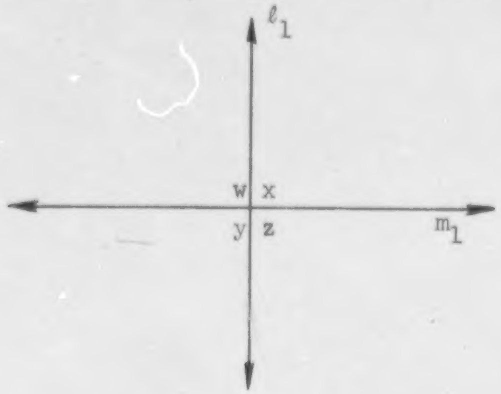
(c) RSTV $\cong$ ________ (g) RSTV $\cong$ ________

(d) RSTV $\cong$ ________ (h) RSTV $\cong$ ________

Work Sheet

Right Angles and Perpendicular Lines

In an earlier chapter you learned that if two lines intersect and the four angles formed are all congruent, then the lines are perpendicular to each other.



The exercises that follow will give you a chance to review some of the ideas of perpendicularity.

Exercises

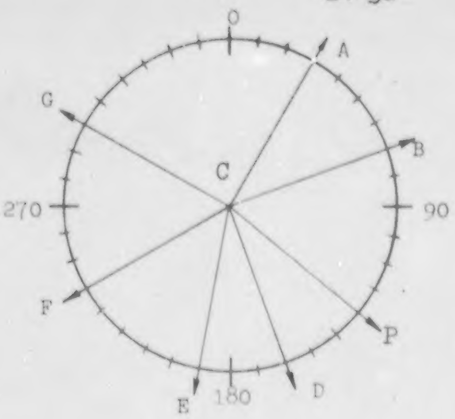
1. (a) To the right, make a sketch showing that a ray pointing north and a ray pointing east are perpendicular to each other, thus forming a right angle.

   (b) A ray pointing north is also perpendicular to a ray pointing in what other direction? _______________

   (c) A ray pointing east is also perpendicular to a ray pointing in what other direction? _______________

2. The circle to the right is marked every 10 degrees. One pair of perpendicular rays is $\overrightarrow{CA}$ and $\overrightarrow{CG}$. Name another pair of perpendicular rays:

________ and ________ .

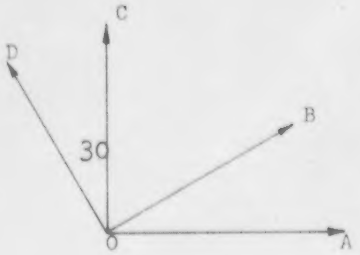


3. ∠ AOC and ∠ BOD are right angles, and m ∠ COD = 30 .

(a) m ∠ COB = ________

(b) m ∠ AOB = ________

(c) m ∠ DOA = ________

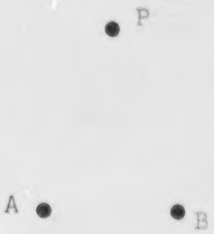


BRAINBOOSTER.

4. Take a piece of paper and try to fold and crease it twice so that when you open it back up the creases will be perpendicular to each other.

Sets of Equidistant Points in a Plane (Perpendicular Bisectors)

In the drawing below the distance from point P to point A is the same as the distance from point P to point B. Therefore P is said to be equidistant from A and B.



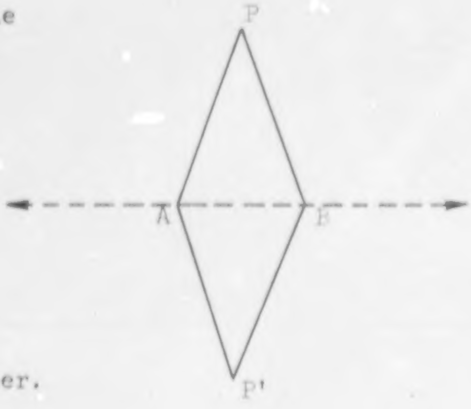
Class Discussion

1. Use a straightedge to draw $\overline{AP}$, $\overline{PB}$, and $\overline{AB}$.

2. What kind of figure has been formed? ____________________

3. This particular type of triangle is called an isosceles triangle.
   (a) What do you know about m $\overline{PA}$ and m $\overline{PB}$?
   ____
   (b) If two sides of a triangle are congruent, then the triangle is an ____________________ triangle.

4. Suppose we take the isosceles triangle $\triangle APB$ and, using the line that passes through point A and B as a flip axis, we flip over the triangle so that we form a figure like the one on the right.



   (a) Is $\overline{PA} \cong \overline{P'A}$? __________
   (b) Is $\overline{P'A} \cong \overline{P'B}$? __________
   (c) Is $\overline{P'B} \cong \overline{BP}$? __________
   (d) Is $\overline{BP} \cong \overline{AP}$? __________
   (e) Then $\overline{PA}$, $\overline{P'A}$, $\overline{P'B}$ and $\overline{BP}$ are all _______________ to each other.
   (f) What kind of figure is APBP'?

5. In the figure of Problem 4, draw the diagonal $\overline{PP'}$ and label the point where the diagonals intersect, M . By using the SSS property of congruence, we can show that $\triangle APM \cong \triangle BPM$. See if you can follow this reasoning:

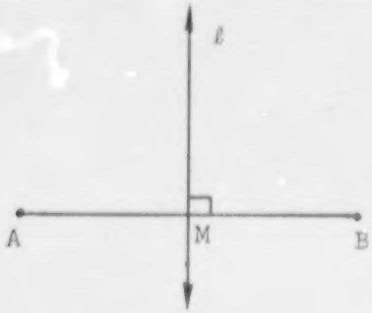
(1) $\overline{AP} \cong \overline{BP}$ This was a given fact.
(2) $\overline{MA} \cong \overline{MB}$ The diagonals of a rhombus bisect each other.
(3) $\overline{PM} \cong \overline{PM}$ Identity congruence.

Then, by the SSS property of congruence,

$$\triangle APM \cong \triangle BPM .$$

Exercises

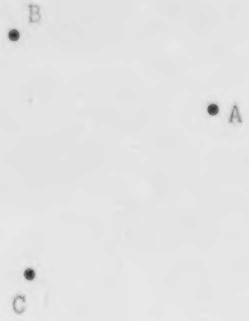
1. Below is a line segment $\overline{AB}$ and its perpendicular bisector ℓ .



(a) Pick any point on ℓ above $\overline{AB}$ and label it P .
(b) Draw $\overline{PA}$ and $\overline{PB}$.
(c) Show that any point P that lies on ℓ is equidistant from A and B . To do this, use the SAS congruence property.
   (1) $\overline{MA} \cong \overline{MB}$ because ______________________________.
   (2) $\angle AMP \cong \angle BMP$ because ______________________________.
   (3) $\overline{MP} \cong \overline{MP}$ because ______________________________.
   (4) Therefore, $\triangle APM \cong \triangle$ ______.

Now, since point P was any point on ℓ , this tells you that each point on the perpendicular bisector ℓ is equidistant from A and B .

2. You are given points A , B , and C .



(a) Draw $\overline{BC}$ and $\overline{AC}$.

(b) Construct the perpendicular bisectors of $\overline{BC}$ and $\overline{AC}$ such that they intersect.

(c) Label the point of intersection of the perpendicular bisectors, D .

(d) Are points A , B , and C equidistant from point D ?

____

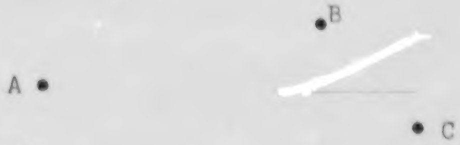
(e) Place the needlepoint of your compass on point D and the pencil point on point A .

(f) Now draw a circle.

(g) Do the points B and C lie on the circle? __________

(h) If you are given any three points of a circle, can you always find the center of the circle? ______________

3. Points A , B , and C are all equidistant from a certain point.

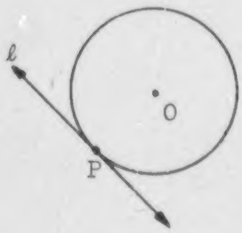


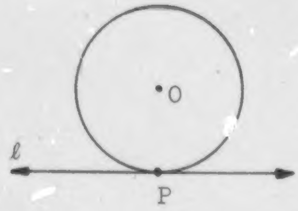
(a) Find that certain point. Label it O .

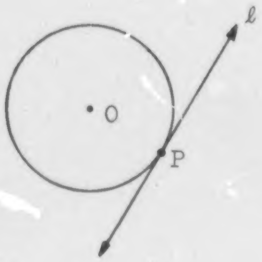
(b) Make a drawing that shows all the points equidistant from point O .

(c) What is this figure called? ____________________

Circles and Perpendiculars







Think of the drawings above as representing a wheel going downhill, on level ground, and then uphill.

In each case, line ℓ touches the circle at exactly one point and is therefore called a tangent. Point P is called the point of contact or point of tangency.

Class Discussion

1. For each of the circles above, draw $\overline{OP}$.

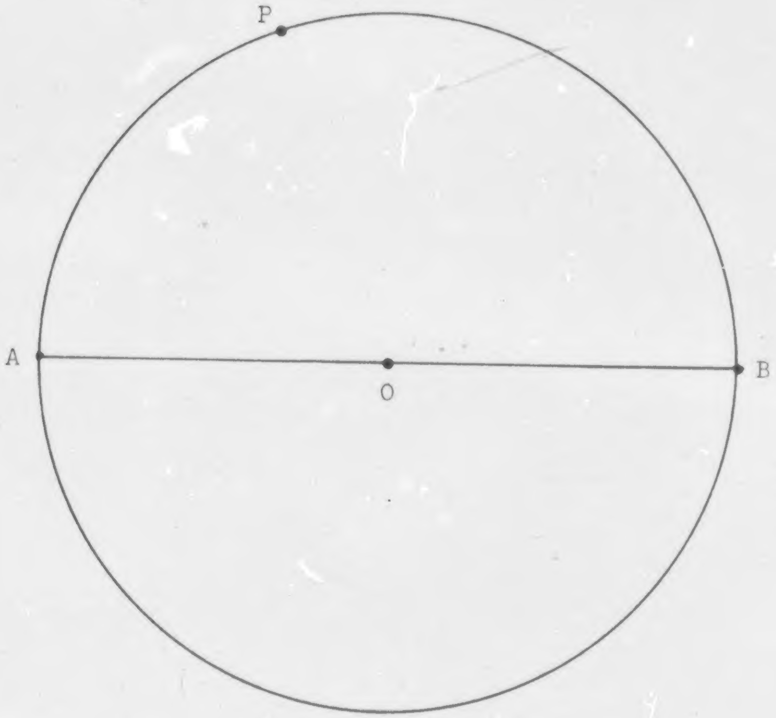
2. (a) What do you think is the relation between the tangent line ℓ and the radius $\overline{OP}$? ____________________

   (b) Mathematicians can prove that:

      (i) A line tangent to a circle is perpendicular to a radius at the point of tangency.

      (ii) A line perpendicular to a radius at its outer endpoint is tangent to the circle.

3. Pictured below is a circle with center O and diameter $\overline{AB}$. (A line segment through the center of a circle is the diameter if its endpoints lie on the circle.) Point P is a point on the circle.



(a) Draw $\overrightarrow{AP}$. Be sure the ray extends past point P.

(b) Use your compass and straightedge to construct a line perpendicular to $\overrightarrow{AP}$ passing through point P.

(c) Does the line you just constructed also pass through point B? __________ It should.

(d) What kind of an angle is ∠ APB? ____________________

(e) What kind of a triangle is △ APB? ____________________

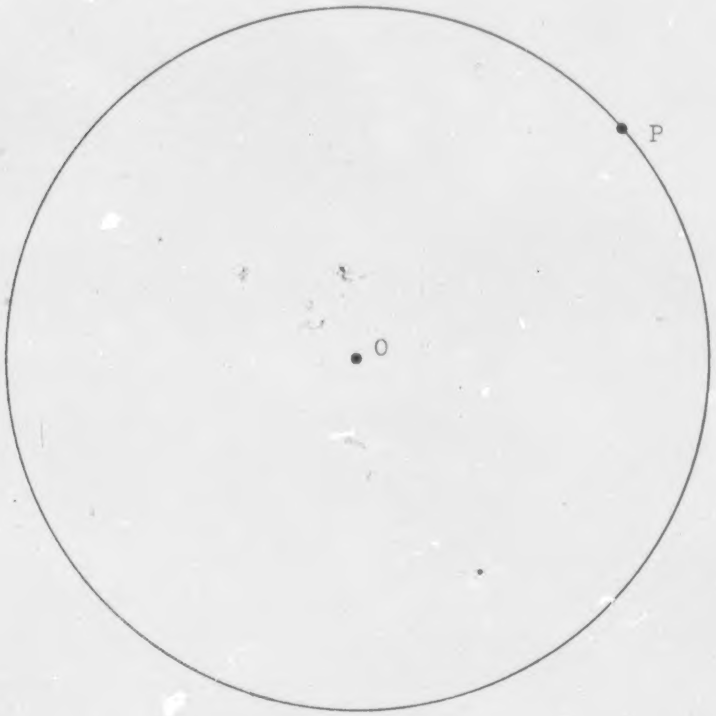
If you were to do the same construction for any other point on the semi-circle, the result would be the same.

Any angle formed by two rays which have a common vertex on a circle and which go through the endpoints of a diameter of that circle is a right angle.

Exercises

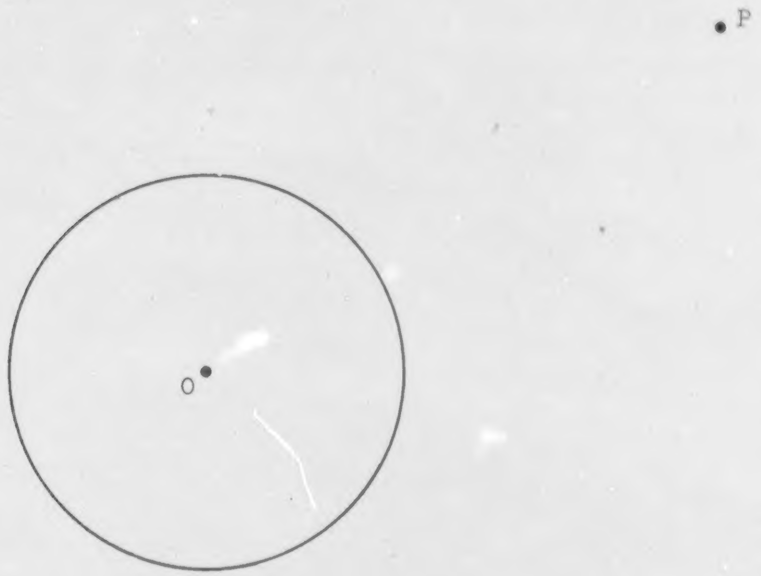
1. (a) In the circle below draw ray $\overrightarrow{OP}$.

   (b) Construct a line ℓ perpendicular to ray $\overrightarrow{OP}$ passing through point P .



   (c) You know that the line ℓ you just constructed is perpendicular to the radius $\overline{OP}$: therefore, line ℓ is ________________ to the circle.

2. You will now construct two tangents from point P to the circle drawn below.



(a) Draw $\overline{OP}$.

(b) Find the midpoint M of $\overline{OP}$ by bisecting $\overline{OP}$.

(c) Put the needle point of your compass on point M and the pencil point on point P .

(d) Now draw a circle intersecting the other circle in two points.

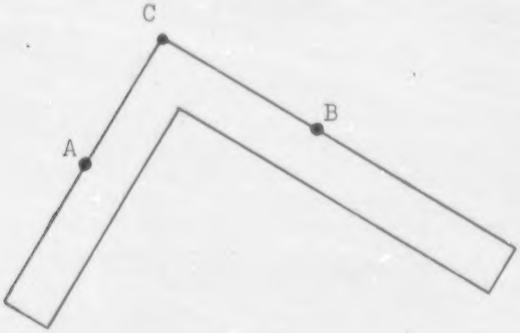
(e) Label these points of intersection Q and R .

(f) Draw $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$. Both of these lines are tangent to the circle at points Q and R.

BRAINBOOSTER.

3. Two nails are driven in at A and B, and a carpenter's square is pressed against the nails as shown.

   Describe the path of point C as the carpenter's square is moved around but kept pressed against the two nails.

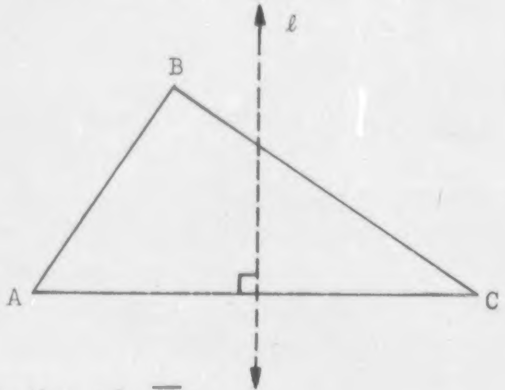


Triangles and Paper Folding

Class Discussion

1. Take Page 14-6c out of your notebook and place it on your desk.

Fold the paper so that C falls on top of A, and crease the paper along the dotted line ℓ.



Line ℓ is the perpendicular bisector of $\overline{AC}$.

Now fold A on top of B, and crease the paper to show the perpendicular bisector of $\overline{AB}$.

Now fold B on top of C, and crease the paper to show the perpendicular bisector of $\overline{BC}$.

(a) Use a straightedge and draw the lines represented by the creases.

(b) Look at the three perpendicular bisectors of the three sides of the triangle. Do they intersect at a point? ________

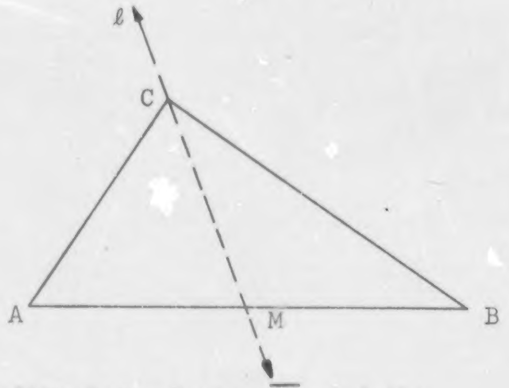
(c) Place the needle point of your compass at the point of intersection and the pencil point on vertex A. Now draw a circle. Do the points B and C lie on the circle you drew? ________

(d) Is the point of intersection of the three perpendicular bisectors of the three sides of the triangle equidistant from each vertex? ________

2. Take Page 14-6d out of your notebook. Fold B on A and pinch the paper at M to mark the midpoint of $\overline{AB}$.

   Then fold the paper so that the crease passes through C and M as shown by the dotted line ℓ.

   $\overline{CM}$ is called a median of the triangle.



   Using the same method, find the median from B to $\overline{AC}$ and the median from A to $\overline{BC}$.

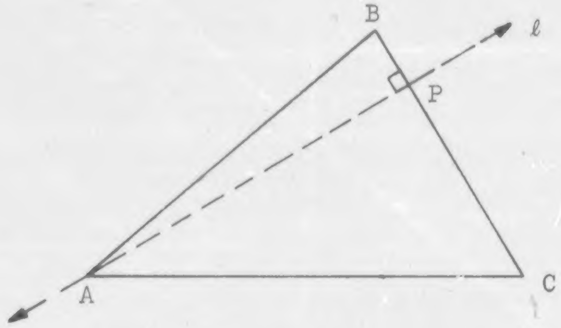
   (a) Use a straightedge and draw the lines represented by the creases.

   (b) Do the three medians intersect in a point? __________

3. Take Page 14-6e out of your notebook. Fold the paper so that B falls on $\overline{BC}$ and the crease passes through A as shown.

   $\overline{AP}$ is called an altitude of the triangle.

   Use the same procedure to find the altitude from B to $\overline{AC}$ and the altitude from C to $\overline{AB}$.



   (a) Use a straightedge and draw the lines represented by the creases.

   (b) Do the three altitudes intersect at a point? __________

   (c) Is each altitude perpendicular to a side of the triangle?

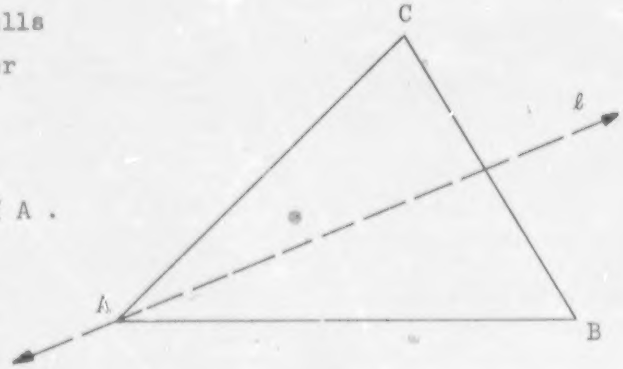
   ____

4. Take Page 14-6f out of your notebook.

Fold the paper so that $\overline{AB}$ falls along $\overline{AC}$ and crease the paper along the dotted line ℓ as shown.

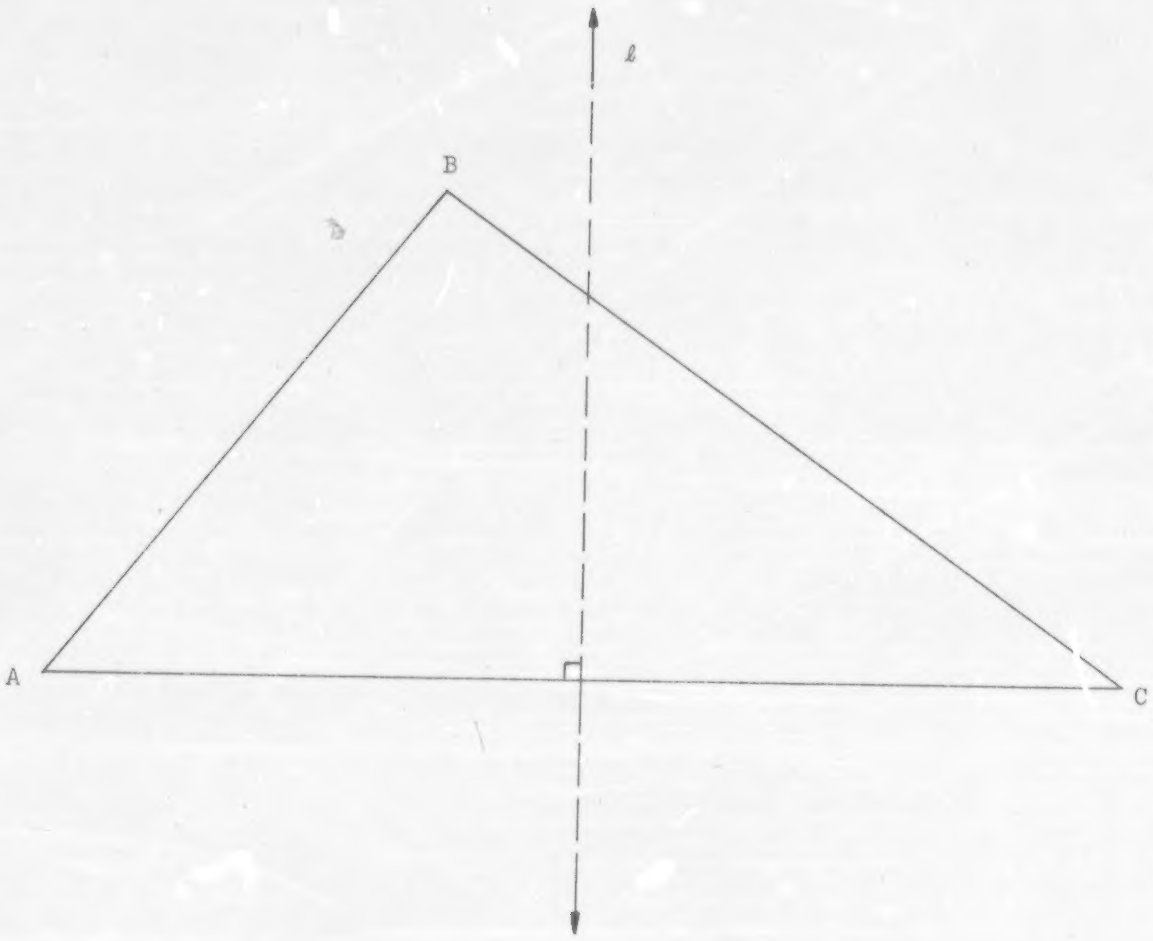
Line ℓ is the bisector of $\angle A$.

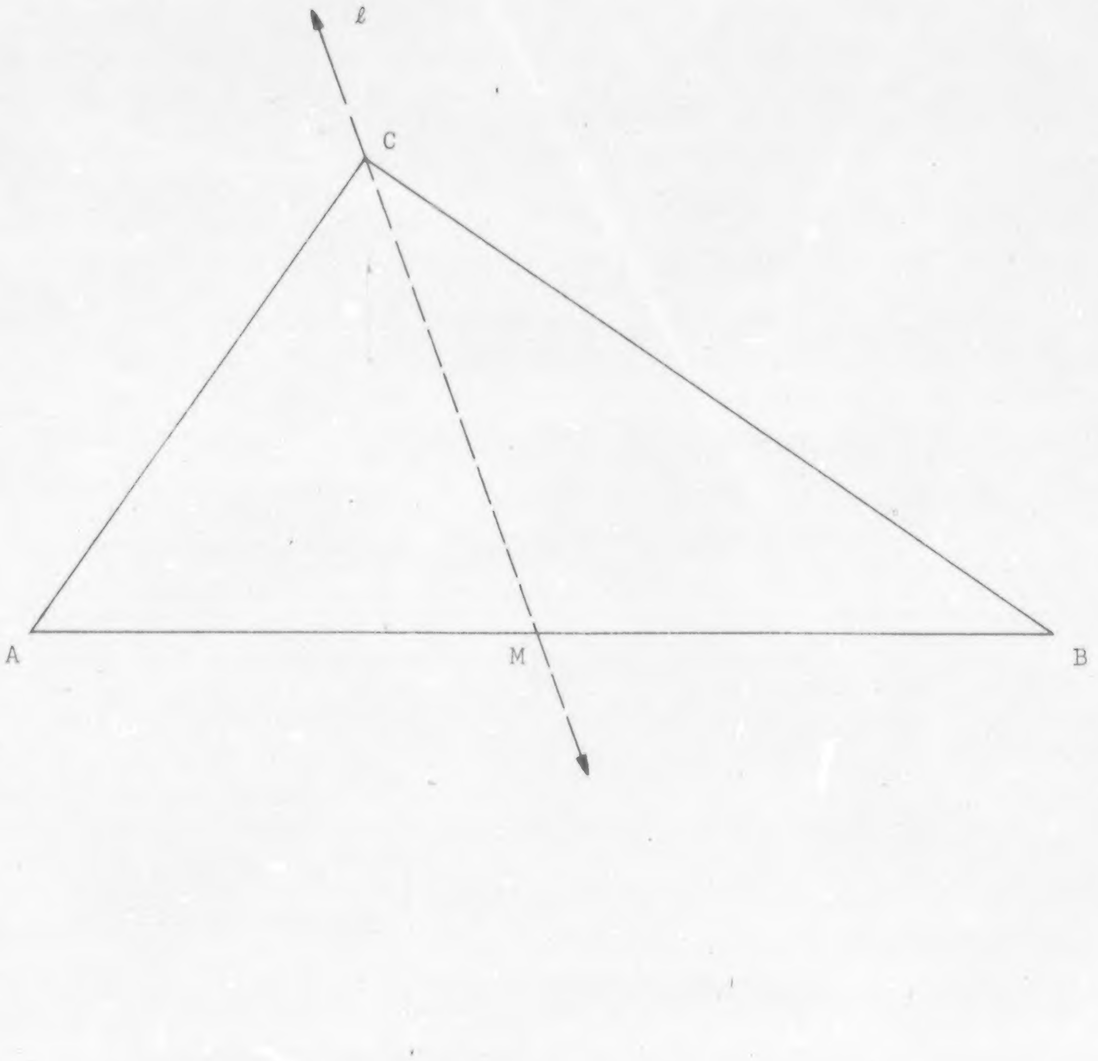
In the same way fold and crease the paper to show the bisectors of $\angle B$ and $\angle C$.

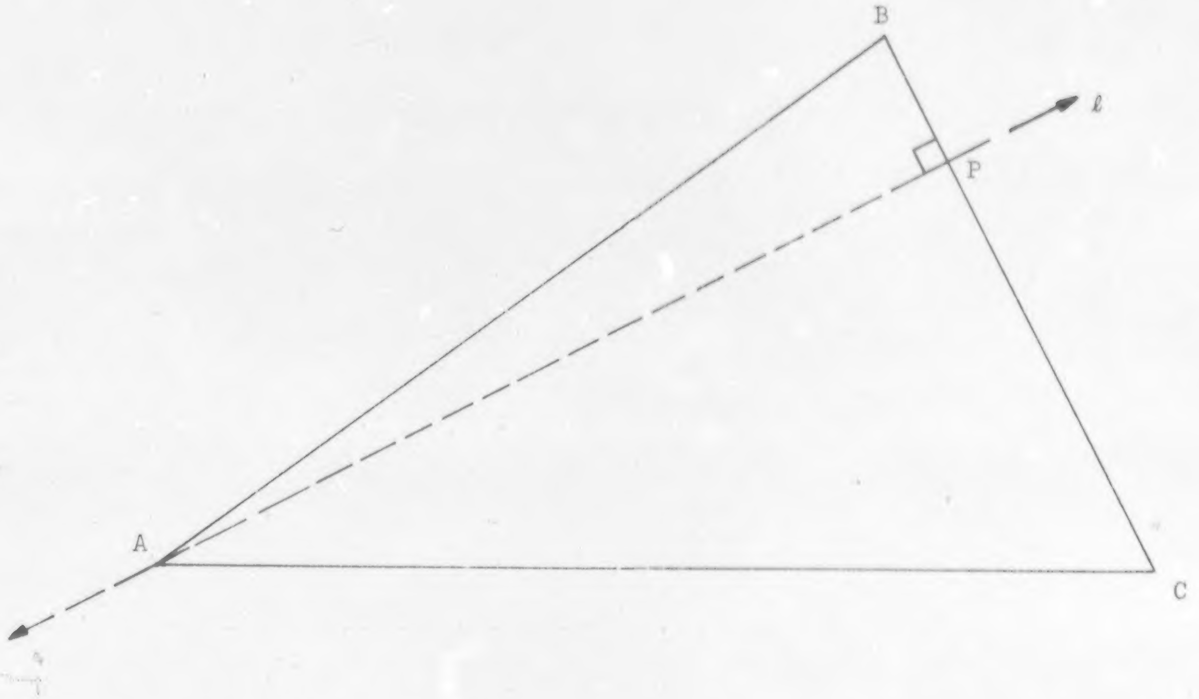


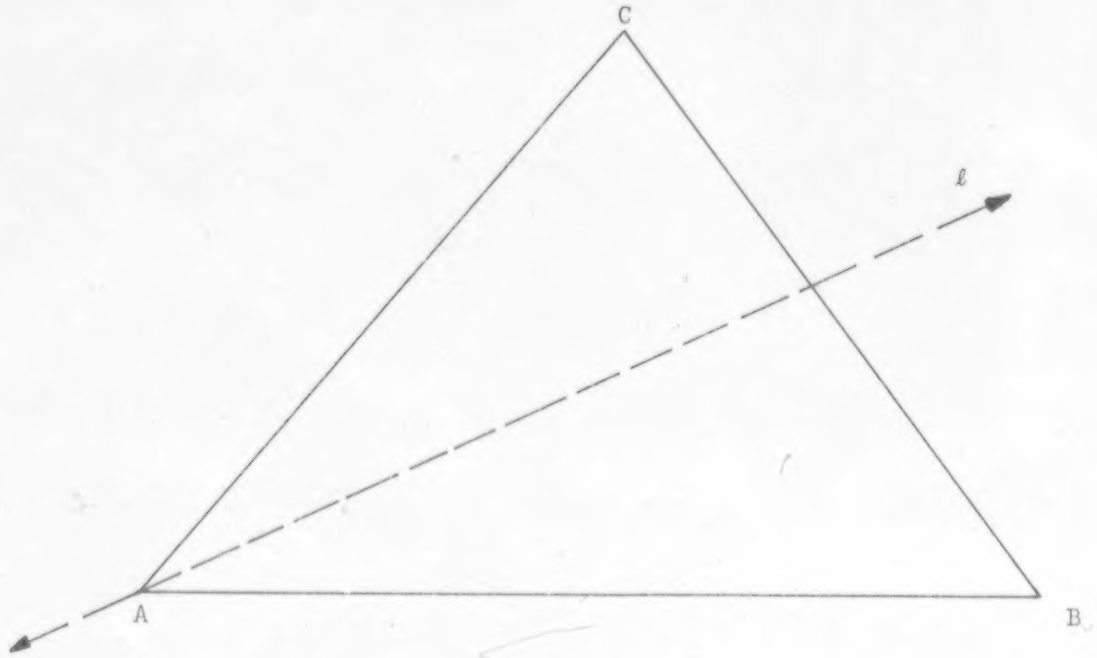
(a) Use a straightedge and draw the lines represented by the creases.

(b) Do the three angle bisectors intersect at a point? ________


ℓ
B
A
C

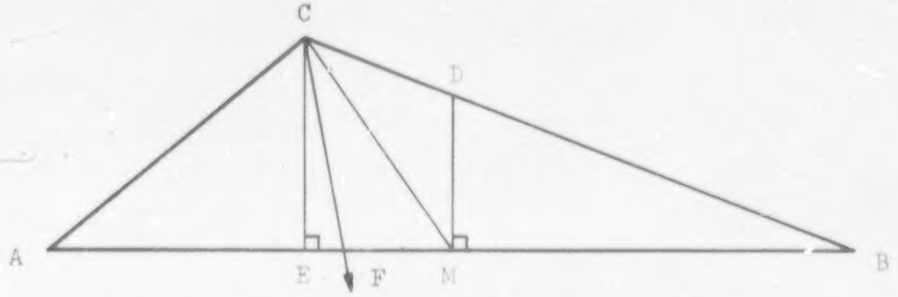

ℓ
C
A
M
B


B
ℓ
P
A
C


C
ℓ
A
B

Exercises

1. Use the figure below to answer the questions that follow.
   $\overline{MA} \cong \overline{MB}$ and $\angle ACF \cong \angle BCF$.



Name a segment that shows:

(a) a perpendicular bisector of a side of the triangle. __________

(b) a median of the triangle. __________

(c) an altitude of the triangle. __________

(d) an angle bisector of an angle of the triangle. __________

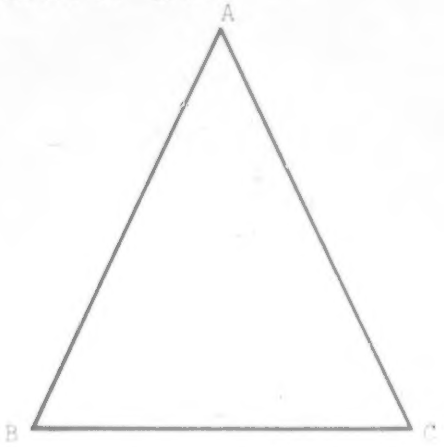
2. ΔABC is an isosceles triangle with $\overline{AB} \cong \overline{AC}$. Use compass and straightedge to construct:

(a) the perpendicular bisector of $\overline{BC}$.

(b) the altitude from A to $\overline{BC}$.

(c) the median from A to $\overline{BC}$.

(d) the bisector of $\angle A$.

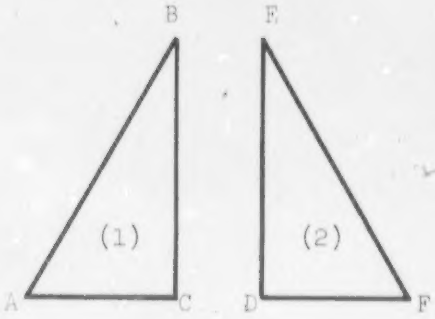


Pre-Test Exercises

These exercises are like the problems that will be on the chapter test. If you don't know how to do them, read the section again. If you still don't understand, ask your teacher.

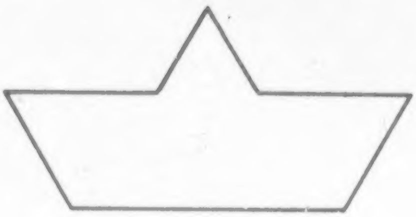
1. (Section 14-1.)

   The figures below are congruent. Using arrows to show slides or turns and dotted lines to show flip axes, show two methods of making triangle (1) map onto triangle (2) .



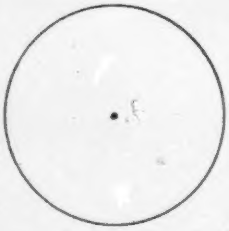
2. (Section 14-1.)

   In the figure below, draw a flip axis such that the figure maps onto itself.



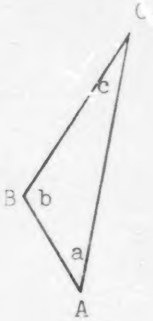
3. (Section 14-1.)

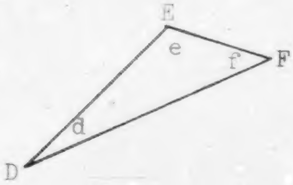
How many flip axes does the figure below have? ________________



4. (Section 14-2.)

The two triangles below are congruent. List the corresponding parts for each triangle.





________ corresponds to ________

________ corresponds to ________

________ corresponds to ________

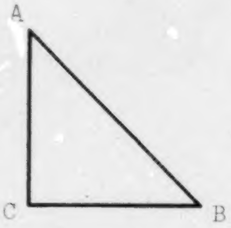
∠ ________ corresponds to ∠ ________

∠ ________ corresponds to ∠ ________

∠ ________ corresponds to ∠ ________

5. (Section 14-2.)

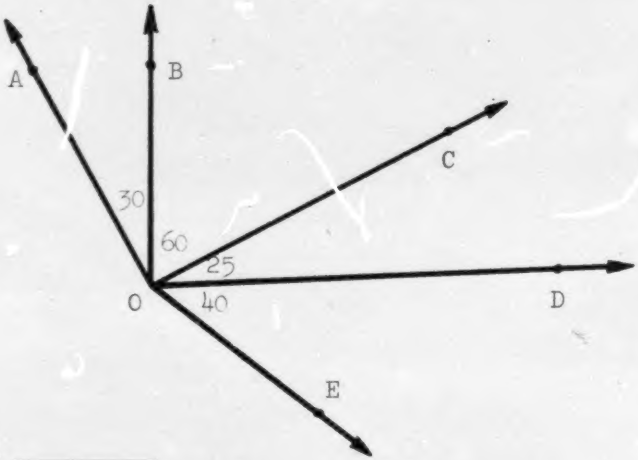
The figure below is an isosceles right triangle where $\overline{CA} \cong \overline{CB}$. Write two correspondences that show that $\triangle ACB$ is congruent to itself.



(a) $\triangle ACB \cong$ ________ (b) $\triangle ACB \cong$ ________

6. (Section 14-3.)

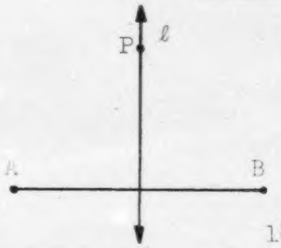
Name a pair of rays that are perpendicular to each other.



________ ⊥ ________

7. (Section 14-4.)

In the figure below line ℓ is the perpendicular bisector of $\overline{AB}$ and point P is on line ℓ . Draw $\overline{PA}$ and $\overline{PB}$. What kind of triangle have you drawn? ____________

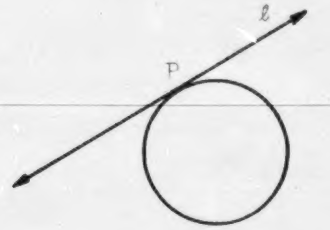


8. (Section 14-4.)

All points on a circle are equidistant from the ______________ of the circle.

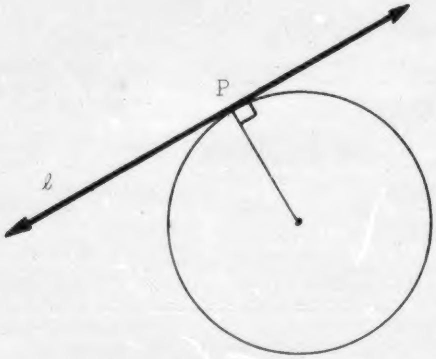
9. (Section 14-5.)

In the drawing below, line ℓ touches the circle in exactly one point, P ; therefore, the line is ______________ to the circle.



10. (Section 14-5.)

The drawing below shows that a line tangent to a circle is ______________________________ to a radius of the circle.



11. (Section 14-6.)

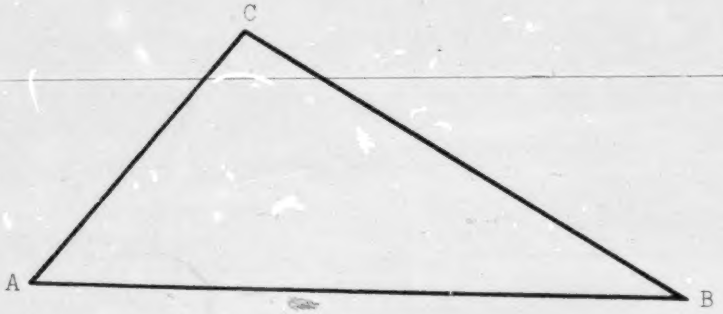
In the triangle below, draw in:

(a) the perpendicular bisector of $\overline{AB}$.

(b) the altitude from C to $\overline{AB}$.

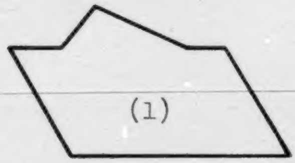
(c) the median from A to $\overline{CB}$.

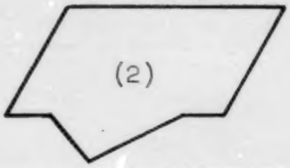
(d) the bisector of $\angle B$.



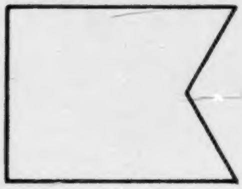
TEST

1. The figures below are congruent. Using arrows to show slides or turns and dotted lines to show flip axes, show two methods of making the figure (1) map onto figure (2).

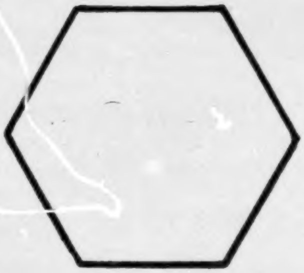




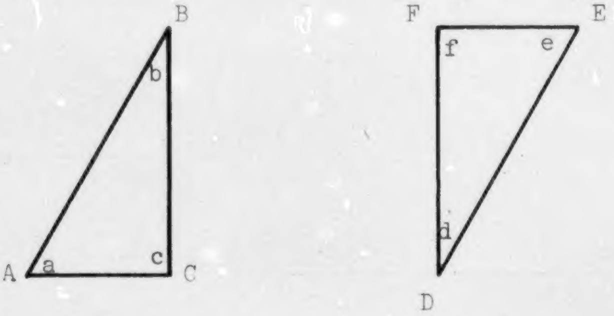
2. In the figure below, draw a flip axis such that the figure maps onto itself.



3. How many flip axes does the figure below have? ____________________



4. The two triangles below are congruent. List the corresponding parts for each triangle.



________ corresponds to ________

________ corresponds to ________

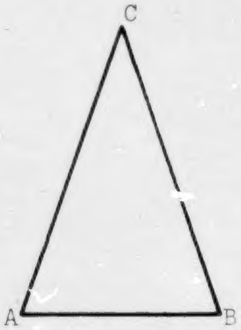
________ corresponds to ________

∠ ________ corresponds to ∠ ________

∠ ________ corresponds to ∠ ________

∠ ________ corresponds to ∠ ________

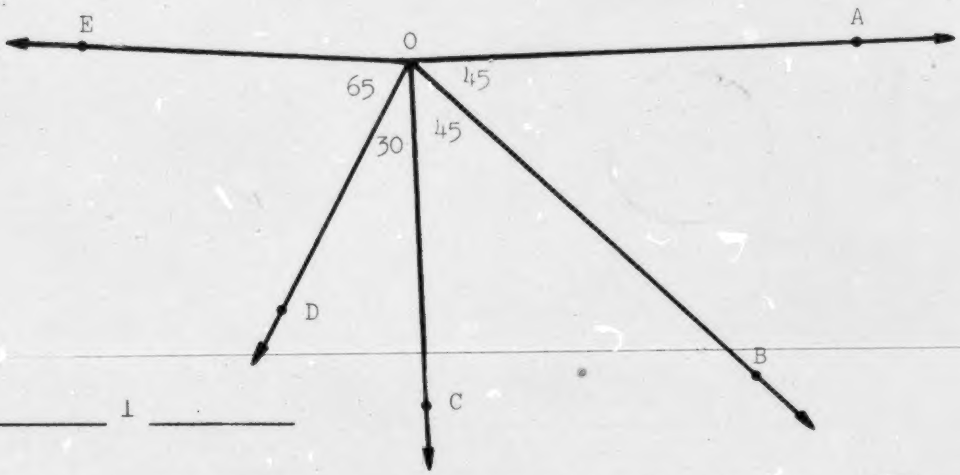
5. The figure below is an isosceles triangle where $\overline{CA} \cong \overline{CB}$. Write two correspondences that show that $\triangle ACB$ is congruent to itself.



(a) $\triangle ACB \cong$ ________

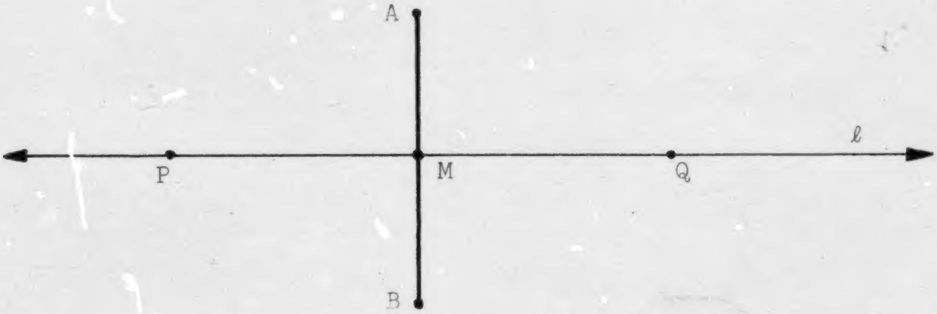
(b) $\triangle ACB \cong$ ________

6. Name a pair of rays that are perpendicular to each other.



_________ ⊥ _________

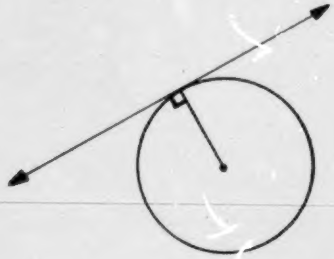
7. In the figure below, line ℓ is the perpendicular bisector of $\overline{AB}$. Points P and Q are on line ℓ and are equidistant from M. Draw $\overline{PA}$, $\overline{PB}$, $\overline{QA}$, and $\overline{QB}$. What kind of a figure have you drawn? ___________________



8. All points on a circle are ________________________________ from the center of the circle.

9. If a line is tangent to a circle, then the line touches the circle in exactly __________ point.

10. The drawing below shows that a line ______________________________ to a circle is perpendicular to a radius of the circle.



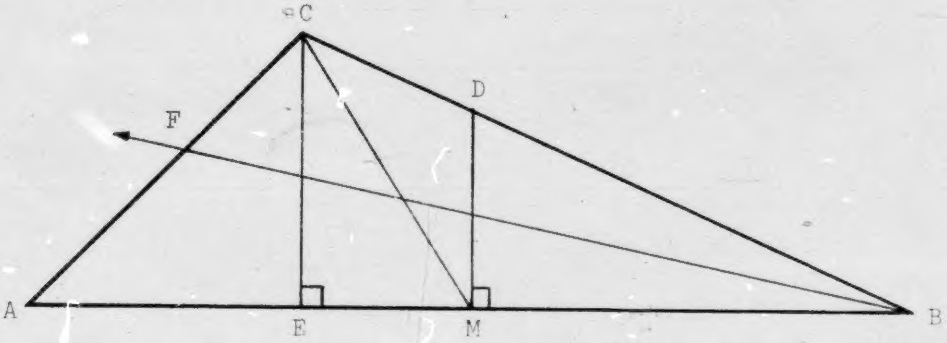
11. In the triangle below $\overline{AM} \cong \overline{BM}$ and $\angle CBF \cong \angle ABF$:

(a) Segment $\overline{DM}$ is called the __________________________ of $\overline{AB}$.

(b) Segment $\overline{CE}$ is called an ____________________________.

(c) Segment $\overline{CM}$ is called a ____________________________.

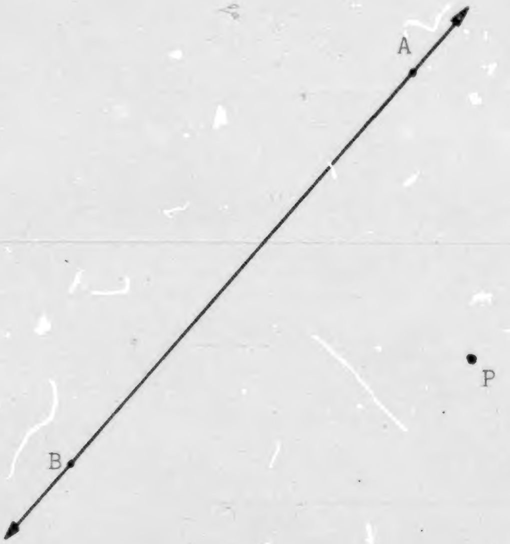
(d) Ray BF __________________ the angle at B .



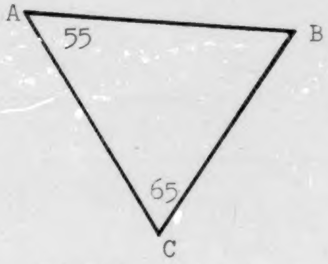
Check Your Memory: Self-Test

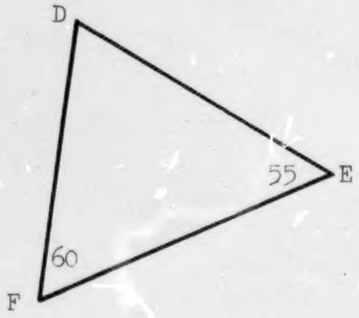
1. (Section 11-3.)

Using a compass and straightedge, construct a line through point P parallel to line $\overleftrightarrow{AB}$.



2. (Sections 12-2 and 11-8.)





Are the triangles shown above similar? __________ If so, what similarity property did you use? ____________________

3. (Section 12-7.)

Use ratios to solve these problems.

(a) 25% of 48 is ______________.

(b) 150% of 60 is ______________.

(c) What percent of 75 is 15 ? ______________

4. (Section 13-2.)

Divide.

(a) $\frac{^-35}{7}$ = ____________

(b) $\frac{108}{^-12}$ = ____________

(c) $\frac{^-45}{^-15}$ = ____________

(d) $\frac{64}{^-16}$ = ____________

(e) $\frac{^-125}{^-25}$ = ____________

5. (Section 13-7.)

Subtract. (Use the right side of the page for your work. Write the answer in simplest form on the blank.)

(a) $\frac{1}{4} - \frac{1}{2}$ = __________

(b) $\frac{3}{5} - \frac{1}{10}$ = __________

(c) $\frac{5}{8} - \frac{1}{3}$ = __________

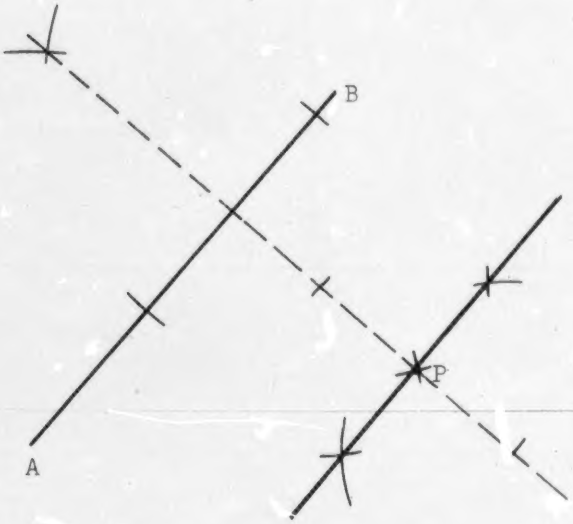
(d) $\frac{9}{4} - \frac{11}{16}$ = __________

(e) $\frac{5}{3} - \frac{5}{6}$ = __________

Now check your answers on the next page. If you do not have them all right, go back and read the section again.

Answers to Check Your Memory: Self-Test

1.



2. Yes. The angle at B is a 60° angle and the angle at D is a 65° angle, so the three angles of one triangle are congruent to the angles of the other.

3. (a) $\frac{25}{100} = \frac{?}{48}$ 25% of 48 is 12.
(b) $\frac{150}{100} = \frac{?}{60}$ 150% of 60 is 90.
(c) $\frac{?}{100} = \frac{15}{75}$ 20% of 75 is 15.

4. (a) $^-5$
(b) $^-9$
(c) 3
(d) $^-4$
(e) 5

5. (a) $\frac{^-1}{4}$
(b) $\frac{5}{10} = \frac{1}{2}$
(c) $\frac{7}{24}$
(d) $\frac{25}{16}$
(e) $\frac{5}{6}$